Midterm 2 November 16, 2016 Duration: 50 minutes

*This test has **4 questions** on **5 pages**, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, unless otherwise indicated.
- Continue on the closest blank page if you run out of space, and **indicate this clearly on the original page**.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: ______________________

Student-No: ______________________ Section: ______________________

Signature: ______________________

Question:	1	2	3	4	Total
Points:	9	9	12	10	40
Score:					

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

9 marks 1. Find all critical points of

$$f(x,y) = 9x^3 + \frac{1}{3}y^3 - 27xy,$$

and classify each critical point as a local maximum, a local minimum, or a saddle point.

Answer: $(0,0)$ is a saddle point,
$(3,9)$ is a local minimum.

Solution: The critical points are the solutions to

$$f_x = 27x^2 - 27y = 0$$
$$f_y = y^2 - 27x = 0.$$

From the first equation we obtain $y = x^2$. We insert this into the second equation to obtain $x^4 - 27x = 0$, so $x(x^3 - 27) = 0$. Thus $x = 0$ or $x = 3$, and the critical points are $(0,0)$ and $(3,9)$.

For their classification, we compute

$$f_{xx} = 54x, \quad f_{yy} = 2y, \quad f_{xy} = -27,$$

and so $D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 2 \cdot 27xy - 27^2$ is

$$D(0,0) = -27 < 0, \quad D(3,9) = 2 \cdot 2 \cdot 27 \cdot 3 \cdot 9 - 27^2 = 4 \cdot 27^2 - 27^2 = 3 \cdot 27^2 > 0.$$

Therefore $(0,0)$ is a saddle point, and since $f_{xx}(3,9) = 54 \cdot 3 > 0$, we obtain that $(3,9)$ is a local minimum.

$\boxed{\text{9 marks}}$ 2. Using Lagrange multipliers, determine the largest volume of a cylinder whose radius r and height h satisfy the equation $r + 2h = 3$.

Answer: 2π

Solution: The volume is $V(r, h) = \pi r^2 h$, and the constraint is

$$g(r, h) = r + 2h = 3.$$

Since $\vec{\nabla} V = \pi\langle 2rh, r^2\rangle$ and $\vec{\nabla} g = \langle 1, 2\rangle$, we have the three equations

$$\begin{aligned} \pi 2rh &= \lambda \\ \pi r^2 &= 2\lambda \\ r + 2h &= 3. \end{aligned}$$

From the first two equations we have $4rh = r^2$, so $r = 4h$ as we may assume $r > 0$. From the third equation we have $6h = 3$, so $h = 1/2$ and $r = 2$. Thus the maximum volume is $V = 2\pi$.

3. Let D be the bounded region in $\mathbb{R}^2$ contained between the curves $x = y^2$ and $x^3 = 32y$.

4 marks (a) Express the double integral $\iint_D f(x,y)\,dx\,dy$ in the order $\int_?^? \int_?^? f(x,y)\,dx\,dy$.

Answer:

$$\int_0^2 \int_{y^2}^{32^{1/3}y^{1/3}} f(x,y)\,dx\,dy$$

Solution: The curves $x = y^2$ and $x^3 = 32y$ intersect at $(0,0)$ and $(4,2)$. The region D can be expressed as

$$D = \left\{(x,y) : \, y^2 \le x \le 32^{1/3}y^{1/3}, 0 \le y \le 2\right\}.$$

Therefore the integral is $\int_0^2 \int_{y^2}^{32^{1/3}y^{1/3}} f(x,y)\,dx\,dy$.

4 marks (b) Now express the double integral $\iint_D f(x,y)\,dx\,dy$ in the order $\int_?^? \int_?^? f(x,y)\,dy\,dx$.

Answer:

$$\int_0^4 \int_{x^3/32}^{\sqrt{x}} f(x,y)\,dy\,dx$$

Solution: The region D can also be expressed as

$$D = \left\{(x,y) : \, x^3/32 \le y \le \sqrt{x}, 0 \le x \le 4\right\}.$$

Therefore the integral is $\int_0^4 \int_{x^3/32}^{\sqrt{x}} f(x,y)\,dy\,dx$.

4 marks (c) Let $f(x,y) = \cos\left(\frac{2}{3}x^{3/2} - \frac{x^4}{128}\right)$. Compute the integral $\iint_D f(x,y)\,dx\,dy$ (in any order of your choice).

Answer: $\sin(10/3)$

Solution: The integral is

$$\begin{aligned}\int_0^4 \int_{x^3/32}^{\sqrt{x}} \cos\left(\frac{2}{3}x^{3/2} - \frac{x^4}{128}\right) dy\,dx &= \int_0^4 \cos\left(\frac{2}{3}x^{3/2} - \frac{x^4}{128}\right)\left(\sqrt{x} - \frac{x^3}{32}\right) dx \\ &= \int_0^{\frac{16}{3}-2} \cos(t)\,dt = \int_0^{10/3} \cos(t)\,dt \\ &= \sin(10/3).\end{aligned}$$

Here we used the substitution $t = \frac{2}{3}x^{3/2} - \frac{x^4}{128}$ to evaluate the outer integral.

4. Consider $f(x, y) = x^3 y$.

3 marks

(a) Find the directional derivative of f in the direction $\langle 3, 4\rangle$ at the point $(1, 1)$.

Answer: $\frac{13}{5}$

Solution: $\vec{\nabla} f = \langle f_x, f_y\rangle = \langle 3x^2y, x^3\rangle$
$\vec{\nabla} f(1, 1) = \langle 3, 1\rangle$.
$\vec{u} = \frac{1}{5}\langle 3, 4\rangle$. Note: normalized, so $|\vec{u}| = 1$.
$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \frac{9+4}{5} = \frac{13}{5}$.

1 mark

(b) Define a function $G(x, y, z)$ which captures the surface $z = x^3y$ implicitly as its zero level surface.

Answer: $G(x, y, z) = x^3y - z$

Solution: $G(x, y, z) = f(x, y) - z = x^3y - z$.

2 marks

(c) Determine an expression for a normal vector $\vec{n}$ of the surface $z = x^3y$. Your answer should be a function of x and y.

Answer: $\vec{n} = \langle 3x^2y, x^3, -1\rangle$

Solution: $\vec{n} = \vec{\nabla} G(x, y, z)$.

4 marks

(d) Consider a general surface $z = f(x, y)$ with normal vector $\vec{n}$. Let $\vec{v} = \langle f_x(x, y), f_y(x, y), 0\rangle$. Circle T or F to indicate whether each statement is true or false (no further justification is needed).

T **F** $\vec{n}$ points in the direction of maximum increase of f.

T **F** $\vec{n} \times \vec{v}$ is tangential to the surface.

T **F** The equation $\vec{v} \cdot (\langle x, y, z\rangle - \vec{r_0}) = 0$ describes the tangent plane of the surface at $\vec{r_0} = \langle x_0, y_0, f(x_0, y_0)\rangle$.

T **F** The gradient $\vec{\nabla} f$ is orthogonal to the level curves of f.

Solution:
False, $\vec{\nabla} f$ points in the direction (in 2D) in which $f(x, y)$ increases fastest; $\vec{n} = \pm\vec{\nabla} G$ points in the direction (in 3D) in which $G(x, y, z)$ increases/decreases fastest.
True, any vector orthogonal to the normal $\vec{n}$ will be in the tangent plane.
False, expression should be $\vec{n} \cdot (\ldots)$.
True.