

Midterm 2 November 15, 2017 Duration: 50 minutes*This test has 4 questions on 6 pages, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, unless otherwise indicated.
- Continue on the closest blank page if you run out of space, and **indicate this clearly on the original page.**
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

Student Conduct during Examinations

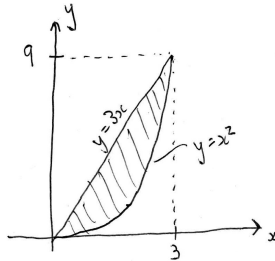
1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

- 3 marks 1. (a) Consider the double integral

$$\int_0^3 \int_{x^2}^{3x} 4x \, dy \, dx.$$

Sketch the region of integration. Label your plot.

Solution:



- 4 marks (b) Switch the limits of integration.

Solution:

$$\int_0^9 \int_{y/3}^{\sqrt{y}} 4x \, dx \, dy$$

- 3 marks (c) Evaluate one of the integrals.

Answer: 27

Solution:

$$\begin{aligned} \int_0^9 \int_{y/3}^{\sqrt{y}} 4x \, dx \, dy &= \int_0^9 2x^2 \Big|_{y/3}^{\sqrt{y}} x \, dx \, dy = \int_0^9 2y - \frac{2y^2}{9} \, dy = y^2 - \frac{2}{3 \cdot 9} y^3 \Big|_0^9 \\ &= 9^2 - \frac{2}{3 \cdot 9} 9^3 = 81 - \frac{2}{3} 9 \cdot 9 = 81 - 2 \cdot 3 \cdot 9 = 81 - 2 \cdot 27 = 27 \end{aligned}$$

or

$$\begin{aligned} \int_0^3 \int_{x^2}^{3x} 4x \, dy \, dx &= \int_0^3 4x [y]_{x^2}^{3x} \, dx = \int_0^3 4x(3x - x^2) \, dx = \int_0^3 12x^2 - 4x^3 \, dx \\ &= [4x^3 - x^4] \Big|_0^3 = 4 \cdot 3^3 - 3^4 = 4 \cdot 27 - 81 = 108 - 81 = 27 \end{aligned}$$

3 marks

2. (a) Suppose we want to minimize $f(x, y, z)$ subject to $g(x, y, z) = k$ for some constant k . State the method of Lagrange multipliers.

Solution:

$$\begin{aligned}\nabla f &= \lambda \nabla g \\ g(x, y, z) &= k\end{aligned}$$

1 mark

- (b) Give an expression $w(x, y, z)$ for the square of the distance from the point (x, y, z) to the origin.

$$\text{Answer: } w(x, y, z) = x^2 + y^2 + z^2$$

5 marks

- (c) Using Lagrange multipliers, find the points on the surface $xy + z = 5$ that are closest to the origin.

$$\text{Answer: } (\pm 2, \pm 2, 1)$$

Solution: This becomes a minimization problem with constraint:

$$\min w = x^2 + y^2 + z^2$$

$$\text{subject to: } xy + z = 5$$

By Lagrange multiplier method, we need to find solutions to

$$\begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ 2z = \lambda \\ xy + z = 5, \end{cases}$$

From the 1st and 2nd equations we find $x^2 = y^2$. So either $x = y$ or $x = -y$.

If $x = y \neq 0$, then $\lambda = 2$ and $z = 1$. So $x^2 = 4$ and $x = y \pm 2$.

If $x = y = 0$, then $z = 5$.

If $x = -y \neq 0$, then $\lambda = -2$ and $z = -1$. So $-x^2 = 6$ which is impossible.

If $x = -y = 0$ then $z = 5$.

We obtain three critical points $(\pm 2, \pm 2, 1), (0, 0, 5)$.

We check: $w(\pm 2, \pm 2, 1) = 4 + 4 + 1 = 9$ and $w(0, 0, 5) = 25$

So the smaller w corresponds to the minimum.

1 mark

- (d) What is that minimum distance?

$$\text{Answer: } \sqrt{9} = 3$$

4 marks

3. (a) Let $f(x, y, z) = xye^{y^2} + e^{z^2}$. Calculate the gradient of f at the point $(1, 1, 1)$ and calculate the directional derivative of $D_{\vec{u}}f$ at the point $(1, 1, 1)$ in the direction of the vector $\vec{v} = \langle 2, -1, 2 \rangle$.

Answer: e **Solution:**

$$\nabla f = \langle ye^{y^2}, xe^{y^2} + 2xy^2e^{y^2}, 2ze^{z^2} \rangle = \langle e, 3e, 2e \rangle = e\langle 1, 3, 2 \rangle.$$

$$\vec{u} = \frac{1}{\sqrt{9}}\langle 2, -1, 2 \rangle, \quad D_{\vec{u}}f(1, 1, 1) = \frac{e}{3}(2 - 3 + 4).$$

6 marks

- (b) Let $g(x, y) = xye^{y^2}$. Find the critical points of g .

Answer: $(0, 0)$ **Solution:** We have already done the derivatives we need. We set

$$\nabla g = \langle ye^{y^2}, xe^{y^2} + 2xy^2e^{y^2} \rangle = \langle ye^{y^2}, (x + 2xy^2)e^{y^2} \rangle = \vec{0}.$$

Critical points: $y = 0$ for first and $x = 0$ or $1 + 2y^2 = 0$ for second. The quadratic has no real roots, so the only common solution is $(0, 0)$. This is the only critical point.

Classify the critical points.

Answer: Saddle at $(0, 0)$

Solution: We find $g_{xx} = 0$, $g_{yy} = 2xy(2y^2 + 3)e^{y^2}$, $g_{xy} = (1 + 2y^2)e^{y^2}$. Evaluating at $(x, y) = (0, 0)$, we have the Hessian matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with determinant $D = 0 \cdot 0 - 1^2 = -1 < 0$; thus we have a saddle.

2 marks

4. (a) A goat walks along a path described by $\vec{r}(t) = \langle x(t), y(t) \rangle$ in the xy -plane. What is her velocity vector at time t ?

$$\text{Answer: } \vec{r}'(t) = \langle x'(t), y'(t) \rangle$$

3 marks

- (b) Consider a mountain with height above the xy -plane given by $z = f(x, y)$. The goat is currently at a point corresponding to (x_0, y_0) . What is her z coordinate?

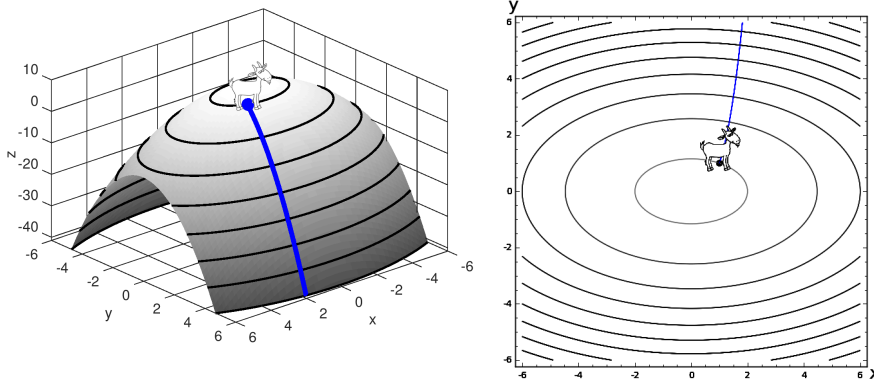
$$\text{Answer: } f(x_0, y_0)$$

In what direction (in the xy -plane) should the goat walk in order to move downhill as quickly as possible?

$$\text{Answer: } -\nabla f(x_0, y_0)$$

5 marks

- (c) Now suppose the mountain height is given by $f(x, y) = 10 - \frac{1}{2}x^2 - \frac{3}{2}y^2$. The goat starts at $t = 0$ at $(1, 1)$ in the xy -plane (with height $z = 8$). Find the goat's path $\vec{r}(t) = \langle x(t), y(t) \rangle$ in the xy -plane, assuming she always takes the steepest path downhill.



$$\text{Answer: } x(t) = e^{ct}, y(t) = e^{3ct} \quad (\text{or } y = x^3)$$

Solution: We need to find $x(t), y(t)$ with $x(0) = 1, y(0) = 1$.

Because the steepest direction gives the fastest way, $\vec{r}'(t)$ will be parallel to $-\nabla f$, and there is some constant c such that $\vec{r}'(t) = -c\nabla f$. This becomes

$$\langle x'(t), y'(t) \rangle = -c \langle -x, -3y \rangle$$

which yields

$$x'(t) = \frac{dx}{dt} = cx, \quad x(0) = 1$$

$$y'(t) = \frac{dy}{dt} = 3cy, \quad y(0) = 1$$

Not many differential equations we remember from last year, but *maybe* these two... We obtain

$$x(t) = e^{ct}, y(t) = e^{3ct}$$

So the path in the xy -plane is $y = x^3$.

Also (not asked for), the surface the path is parametrized by

$$\langle e^{ct}, e^{3ct}, 10 - \frac{1}{2}e^{2ct} - \frac{3}{2}e^{6ct} \rangle$$

or

$$\langle x, x^3, 10 - \frac{1}{2}x^2 - \frac{3}{2}x^6 \rangle$$