Midterm 1 October 11, 2017 Duration: 50 minutes

This test has 4 ***questions*** *on* ***5 pages****, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, unless otherwise indicated.
- Continue on the closest blank page if you run out of space, and **indicate this clearly on the original page.**
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: ____________

Student-No: ____________ Section: ____________

Signature: ____________

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

5 marks 1. (a) Find the area of the triangle with vertices given by the three points $A = (0, 1, 2)$, $B = (1, 0, 3)$, and $C = (1, 1, 4)$.

Answer: $\sqrt{6}/2$

Solution: Use A as a base point and form the vectors $\vec{AB} = \langle 1, 0, 3\rangle - \langle 0, 1, 2\rangle = \langle 1, -1, 1\rangle$ and $\vec{AC} = \langle 1, 1, 4\rangle - \langle 0, 1, 2\rangle = \langle 1, 0, 2\rangle$. The area of the triangle $\triangle ABC$ is then $1/2$ of the length of the vector $\vec{AB} \times \vec{AC}$. Now, $|\vec{AB} \times \vec{AC}| = |\langle -2, -1, 1\rangle| = \sqrt{6}$, so the area of the triangle is $\sqrt{6}/2$.

5 marks (b) Find the angle the plane containing the triangle $\triangle ABC$ makes with the (x, y)-plane. Please leave your answer in "calculator-ready format". -

Answer: $\arccos(1/\sqrt{6})$

Solution: The vector $\vec{n} = \vec{AB} \times \vec{AC}$ is normal to the two vectors $\vec{AB}$ and $\vec{AC}$, and hence is perpendicular to the plane containing the triangle. The (x, y)-plane is $z = 0$ with normal $\hat{k} = \langle 0, 0, 1\rangle$. We can compute the angle θ between these normals using the dot product since $\vec{n} \cdot \hat{k} = |\vec{n}||\hat{k}| \cos(\theta)$. This gives $\langle -2, -1, 1\rangle \cdot \langle 0, 0, 1\rangle = \sqrt{6} \cos(\theta)$, which means $\cos(\theta) = 1\sqrt{6}$, or $\theta = \arccos(1/\sqrt{6})$.

(If you had one of the normals facing the "other way", the dot product would've been negative, in which case the answer could be written in the form $\pi - \arccos(-1/\sqrt{6})$.)

5 marks

2. (a) Let $w(x,y) = \frac{x^2}{x+2y}$. Compute the partial derivatives w_x and w_y.

Answer: $w_x = \frac{x^2+4xy}{(x+2y)^2}$

Answer: $w_y = \frac{-2x^2}{(x+2y)^2}$

Solution: Partial differentiation gives

$$w_x(x,y) = \frac{\left(\frac{\partial}{\partial x}x^2\right)(x+2y) - x^2\left(\frac{\partial}{\partial x}(x+2y)\right)}{(x+2y)^2}$$
$$= \frac{2x(x+2y)-x^2}{(x+2y)^2} = \frac{x^2+4xy}{(x+2y)^2},$$
$$w_y(x,y) = -x^2(x+2y)^{-2}(2) = \frac{-2x^2}{(x+2y)^2}.$$

Can w be a solution of the differential equation $2w_x - w_y = \frac{cx}{x+2y}$?

- [ ] No, not possible.
- [ ] Yes, for any value of c.
- [x] Yes, but only for $c = \boxed{4}$.

Solution: Numerators become: $2x^2+8xy-(-2x^2) = 4x^2+8xy = 4x(x+2y)$ So this cancels one power in the denominator and we're left with $\frac{4x}{x+2y}$.

5 marks

(b) Suppose u is a function of two independent variables x and y. Specifically, let $u(x,y) = x^3y^2$. What is $u_x(1,2)$?

Answer: $u_x = 3x^2y^2$, so $u_x(1,2) = 12$

Suppose further that v is also a function of x and y and $\ln v = u - 3x$. (Here "ln" means the natural logarithm, log base e.) What is $v_x(1,2)$?

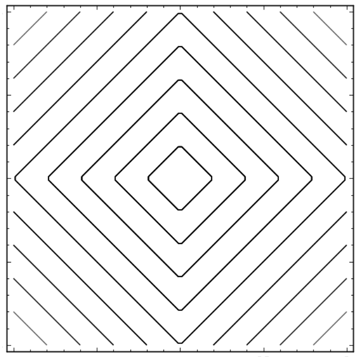
Answer: $9e$

Solution: Implicit differentiation gives $\frac{1}{v}v_x = u_x - 3$, where note $\frac{\partial u}{\partial x}$ is not automatically zero. We need $v(1,2)$ so use $\ln v(1,2) = u(1,2) - 3\cdot 1 = 4-3 = 1$, from which $v(1,2) = e$. Thus we have $\frac{1}{e}v_x(1,2) = u_x(1,2) - 3 = 12-3 = 9$, or $v_x(1,2) = 9e$.

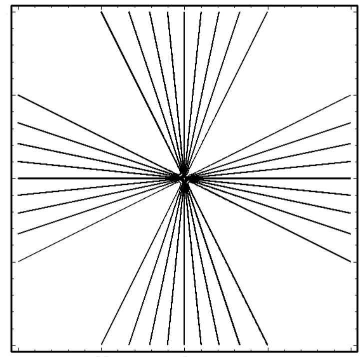
(One might sub it all in, then differentiate, probably more work.)

10 marks

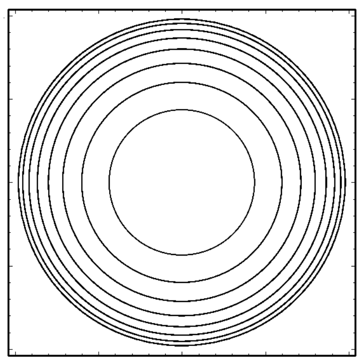
3. Consider the following contour plots, where as usual, the contours are evenly spaced in the z direction.



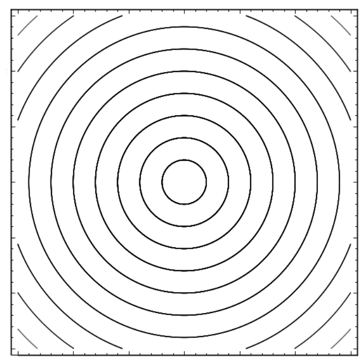
Answer: C



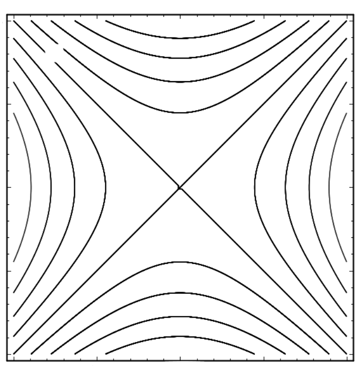
Answer: B



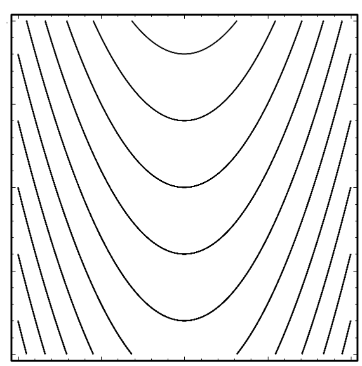
Answer: D, E



Answer: H



Answer: A



Answer: G

In the box below each plot, list the name(s) of *all* possible functions which could generate that contour plot. You do not need to justify your answers.

$$A(x,y) = x^2 - y^2, \quad D(x,y) = \sqrt{1-x^2-y^2}, \quad G(x,y) = y - 2x^2,$$

$$B(x,y) = \frac{xy}{x^2+y^2}, \quad E(x,y) = \sqrt{1-x^2-y^2} + \frac{\sin(10x)}{e^{100}}, \quad H(x,y) = \sqrt{x^2+y^2},$$

$$C(x,y) = |x| + |y|, \quad F(x,y) = x^2 + y^2, \quad I(x,y) = \max(x,y).$$

4. We consider the surface given by the equation $z = x^4 - y^2$. We ask a non-engineer to find the equation of the tangent plane to this surface at the point $(x_0, y_0, z_0) = (2, 3, 7)$. The response is:

$$z = 4x^3\,(x-2) - 2y\,(y-3)$$

2 marks (a) Without any calculation, explain why this cannot be the correct answer.

Solution: The equation of the tangent plane must be a linear equation.

3 marks (b) Find the correct answer.

Answer: $z = 7 + 32\,(x-2) - 6\,(y-3)$

Solution: The equation of the tangent plane to the surface at the point (x_0, y_0, z_0) is $f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\,(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\,(y-y_0)$.
The partial derivatives are $\frac{\partial f}{\partial x} = 4x^3$ and $\frac{\partial f}{\partial y} = -2y$. Injecting $x_0 = 2, y_0 = 3$ yields the answer.

2 marks (c) Give an expression for the total differential $\mathrm{d}z$ (at any point (x,y)).

Answer: $\mathrm{d}z = 4x^3\,\mathrm{d}x - 2y\,\mathrm{d}y$

Solution: In general $\mathrm{d}z = \frac{\partial f}{\partial x}(x,y)\mathrm{d}x + \frac{\partial f}{\partial y}(x,y)\mathrm{d}y$. From above, we have already worked out the partial derivatives. So substituting in (but not evaluating at a point) gives the result.

3 marks (d) At time $t = 0$, an ant leaves the point $(x_0, y_0, z_0) = (2, 3, 7)$ and walks around on the surface, returning to $(2, 3, 7)$ at $t = 2\pi$. The ant's path, when viewed from above, appears to be a unit circle centered at $(x, y) = (1, 3)$ in the xy-plane. Give an expression for the ant's position as a function of time.

Answer: $\vec{l}(t) =$
$\langle \cos t + 1, \sin t + 3, (\cos t + 1)^4 - (\sin t + 3)^2 \rangle$

Solution: It must be a vector. $\vec{l}(t) = \langle x(t), y(t), z(t) \rangle = \langle x(t), y(t), f\,(x(t), y(t)) \rangle$.