Final Examination — June 26, 2018 **Duration: 2.5 hours**

*This test has **8 questions** on **12 pages**, for a total of 100 points.*

Sebastián Barbieri

- Read all the questions carefully before starting to work. Unless otherwise indicated, give complete arguments and explanations for all your calculations as answers without justification will not be marked.
- Continue on a blank page if you run out of space, and **indicate this clearly on the original page.**
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: ____________

Student-No: ____________ Section: ____________

Signature: ____________

Question:	1	2	3	4	5	6	7	8	Total
Points:	20	10	10	10	10	10	10	20	100
Score:									

20 marks

1. There are twelve statements below which you may classify as true or false by marking the corresponding box with an X. The rules are as follows:

- Every correct answer gives you 2 points.
- Every wrong answer gives you -1 point.
- Every answer left blank gives you 0 points.
- You may not earn less than 0 points nor more than 20 points.
- Assume that the functions f, G referenced below are smooth enough (existence of second partial derivatives, equivalence of f_{xy} and f_{yx}, well defined level curves, existence of implicit functions, etc.)

	1	2	3	4	5	6	7	8	9	10	11	12
True	X	X			X			X		X		
False			X	X		X	X		X		X	X

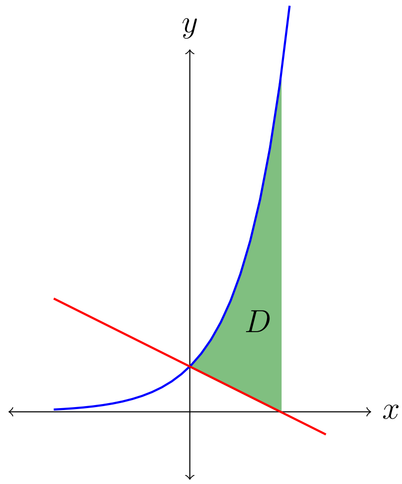
1. For any two (non-zero) non-parallel vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^3$, $||\vec{u} \times \vec{v}|| > 0$.
2. If a continuous function $f(x, y)$ is positive for every (x, y) in its domain D then $\iint_D f(x, y)\mathrm{d}A \geq 0$.
3. If a point (a, b) in the domain of a function $f(x, y)$ satisfies
$$f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)f_{yx}(a, b) = -3$$
then (a, b) is a saddle point.
4. If $\lim_{x\to 0} g(x, mx) = L$ for every value of m and $\lim_{y\to 0} g(0, y) = L$ then the limit $\lim_{(x,y)\to(0,0)} g(x, y)$ exists and is equal to L.
5. $||\vec{u} + \vec{v}|| = ||\vec{u}|| + ||\vec{v}||$ if and only if there exists $a \geq 0$ such that $\vec{u} = a\vec{v}$ or $\vec{v} = a\vec{u}$.
6. Every solution to the equations given by the method of Lagrange multipliers is a local minimum or local maximum of the associated problem.
7. For any three vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$ we have $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.
8. Consider the surface defined by $G(x, y, z) = 0$. If $G(a, b, c) = 0$ then any vector in the tangent plane to the surface at (a, b, c) is orthogonal to $\nabla G(a, b, c)$.
9. Let $z = f(x, y)$ and $c = f(a, b)$. The gradient $\nabla f(a, b)$ is parallel to the curve described by $f(x, y) = c$ at (a, b).
10. Let $z = f(x, y)$. It is possible that $\nabla f(x, y) = (0, 0)$ and (x, y) is neither a minimum nor a maximum.
11. The surface described by $\frac{x^2}{a} + y^2 + z^2 = 1$ is an ellipse for all non-zero values of a.
12. If there are values a, b such that every contour of a function $z = f(x, y)$ is of the form $ax + by = c$ for some value of c then necessarily $z = f(x, y)$ describes a plane.

2. Consider the integral $I = \int_0^2 \int_{1-\frac{x}{2}}^{e^x} f(x,y)\, \mathrm{d}y\mathrm{d}x$

2 marks

(a) Sketch the region D of integration.

Solution:



4 marks

(b) In the identity below the order of integration has been changed. Fill in the blanks.

$$\int_0^2 \int_{1-\frac{x}{2}}^{e^x} f(x,y)\mathrm{d}y\mathrm{d}x = \int_0^1 \int_{\boxed{2-2y}}^{2} f(x,y)\ \mathrm{d}x\mathrm{d}y + \int_1^{\boxed{e^2}} \int_{\boxed{\ln(y)}}^{\boxed{2}} f(x,y)\ \mathrm{d}x\mathrm{d}y$$

4 marks

(c) Compute the value of I for $f(x,y) = y$.

Solution:

$$\begin{aligned}\int_0^2 \int_{1-\frac{x}{2}}^{e^x} y\ \mathrm{d}y\mathrm{d}x &= \int_0^2 \left(\frac{y^2}{2}\right)_{1-\frac{x}{2}}^{e^x}\ \mathrm{d}x \\ &= \frac{1}{2}\int_0^2 e^{2x} - (1 - x + \frac{x^2}{4})\ \mathrm{d}x \\ &= \frac{1}{2}\left(\frac{e^{2x}}{2} - x + \frac{x^2}{2} - \frac{x^3}{12}\right)_0^2 \\ &= \frac{1}{2}\left(\frac{e^4}{2} - 2 + \frac{4}{2} - \frac{8}{12} - \frac{1}{2}\right) = \frac{e^4}{4} - \frac{7}{12}.\end{aligned}$$

Answer: $\frac{e^4}{4} - \frac{7}{12}$.

4 marks

3. (a) Evaluate $I = \iiint_E \sqrt{x^2+y^2}\ \mathrm{d}V$ over the region E bounded by $x^2+y^2+z^2=4$. *Hint:* Recall that $\sin^2(\phi) = \frac{1-\cos(2\phi)}{2}$.

Solution: R is a sphere of radius 2. In spherical coordinates we get that $\sqrt{x^2+y^2} = \rho\sin(\phi)$. Therefore

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (\rho\sin(\phi))\rho^2\sin(\phi)\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta \\ &= \left(\int_0^{2\pi} 1\mathrm{d}\theta\right)\cdot\left(\int_0^{\pi}\sin^2(\phi)\mathrm{d}\phi\right)\cdot\left(\int_0^{2}\rho^3\mathrm{d}\rho\right) \\ &= (2\pi)\cdot\left(\int_0^{\pi}\frac{1-\cos(2\phi)}{2}\mathrm{d}\phi\right)\cdot\left(\frac{16}{4}\right) \\ &= 8\pi\left(\frac{\phi}{2}-\frac{\sin(2\phi)}{4}\right)\Big|_0^{\pi} \\ &= 4\pi^2. \end{aligned}$$

Answer: $I = 4\pi^2$.

6 marks

(b) Consider the triple integral

$$J = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{1} f(x,y,z)\ \mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

Fill in the blanks below where the order of integration has been changed.

$$J = \int_{\boxed{-1}}^{\boxed{1}}\int_{\boxed{-\sqrt{1-y^2}}}^{\boxed{\sqrt{1-y^2}}}\int_{\boxed{x^2+y^2}}^{\boxed{1}} f(x,y,z)\ \mathrm{d}z\mathrm{d}x\mathrm{d}y$$

$$J = \int_{\boxed{-1}}^{\boxed{1}}\int_{\boxed{y^2}}^{\boxed{1}}\int_{\boxed{-\sqrt{z-y^2}}}^{\boxed{\sqrt{z-y^2}}} f(x,y,z)\ \mathrm{d}x\mathrm{d}z\mathrm{d}y$$

4. Consider a right circular cone of height $H > 0$ and radius $a > 0$. This cone may be defined as the solid between $z = 0$ and $z = H(1 - \frac{\sqrt{x^2+y^2}}{a})$ in the region D in the xy-plane bounded by $x^2 + y^2 = a^2$.

Suppose that the density of the cone is proportional to the distance from the z-axis. In other words, there exists a constant $k > 0$ such that $\rho(x, y) = k\sqrt{x^2 + y^2}$.

4 marks (a) Compute the total mass M of the cone.

Solution: We wish to compute $\iiint_R \rho(x, y)\mathrm{d}A$ where R is the cone. This looks nice in cylindrical coordinates.

$$\begin{aligned}
M &= \int_0^{2\pi}\int_0^a\int_0^{H(1-\frac{R}{a})} (kR)R \, \mathrm{d}z \, \mathrm{d}R \, \mathrm{d}\theta \\
&= k\left(\int_0^{2\pi} 1\mathrm{d}\theta\right)\cdot\left(\int_0^a\int_0^{H(1-\frac{R}{a})} R^2 \, \mathrm{d}z \, \mathrm{d}R\right) \\
&= 2k\pi\left(\int_0^a R^2 H(1-\frac{R}{a}) \, \mathrm{d}R\right) \\
&= 2k\pi H\left(\frac{R^3}{3}-\frac{R^4}{4a}\right)_0^a \\
&= 2k\pi H\left(\frac{a^3}{3}-\frac{a^3}{4}\right) = \frac{kHa^3\pi}{6}
\end{aligned}$$

Answer: $M = \frac{kHa^3\pi}{6}$.

5 marks (b) Compute the centre of mass of the cone. You may leave your answers in terms of M.

Solution: By symmetry $\bar{x} = \bar{y} = 0$. We just need to compute $\bar{z}$.

$$\begin{aligned}
\bar{z} &= \frac{1}{M}\int_0^{2\pi}\int_0^a\int_0^{H(1-\frac{R}{a})} zkR^2 \, \mathrm{d}z \, \mathrm{d}R \, \mathrm{d}\theta \\
&= \frac{k}{M}\left(\int_0^{2\pi} 1\mathrm{d}\theta\right)\cdot\left(\int_0^a R^2\int_0^{H(1-\frac{R}{a})} z \, \mathrm{d}z \, \mathrm{d}R\right) \\
&= \frac{2k\pi}{M}\left(\int_0^a R^2\left(\frac{z^2}{2}\right)_0^{H(1-\frac{R}{a})} \mathrm{d}R\right) \\
&= \frac{k\pi}{M}\left(\int_0^a R^2H^2(1-2\frac{R}{a}+\frac{R^2}{a^2}) \, \mathrm{d}R\right) \\
&= \frac{k\pi H^2}{M}\left(\frac{R^3}{3}-2\frac{R^4}{4a}+\frac{R^5}{5a^2}\right)_0^a \\
&= \frac{k\pi H^2}{M}\left(\frac{a^3}{3}-\frac{a^3}{2}+\frac{a^3}{5}\right) = \frac{k\pi H^2a^3}{30M} = \frac{H}{5}
\end{aligned}$$

Answer: $\bar{x} = 0$.

Answer: $\bar{y} = 0$.

Answer: $\bar{z} = \frac{H}{5}$ or $\frac{k\pi H^2 a^3}{30M}$.

1 mark

(c) Upon which of the parameters H, a and k of the problem above does the centre of mass $(\bar{x}, \bar{y}, \bar{z})$ depend?

Answer: Only H.

5. A long time ago in a galaxy far, far away, there was a little world inhabited exclusively by beautiful alpacas. Their world was so small and round that it could be modeled by a sphere $x^2 + y^2 + z^2 \leq R^2$.

$\boxed{\text{2 marks}}$ (a) Suppose a surface is given by the equation $z = f(x, y)$ on the domain D. State the integral formula for the surface area.

Answer: $\iint_D \sqrt{1 + ||\nabla f(x, y)||^2}\, \mathrm{d}A$

$\boxed{\text{4 marks}}$ (b) Let $0 \leq c \leq R$ be a fixed number. Compute the total surface area S of the portion of the alpaca planet that is above $z = c$.

Solution: The gradient $\nabla f(x, y)$ is $\left(\frac{-x}{\sqrt{R^2-x^2-y^2}}, \frac{-y}{\sqrt{R^2-x^2-y^2}}\right)$, thus

$$\sqrt{1 + ||\nabla f(x, y)||^2} = \frac{R}{\sqrt{R^2 - x^2 - y^2}}$$

Hence, in polar coordinates we get

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^{\sqrt{R^2-c^2}} \frac{R}{\sqrt{R^2 - \rho^2}}\, \rho \; \mathrm{d}\rho \mathrm{d}\theta \\ &= -R\pi \int_{R^2}^{c^2} \frac{1}{\sqrt{u}}\, \mathrm{d}u \\ &= -R\pi \left(2\sqrt{u}\right)_{R^2}^{c^2} = 2\pi R(R - c). \end{aligned}$$

Where the change of variable was $u = R^2 - \rho^2$ with $\mathrm{d}u = -2\rho \mathrm{d}\rho$.

Answer: $S = 2\pi R(R - c)$

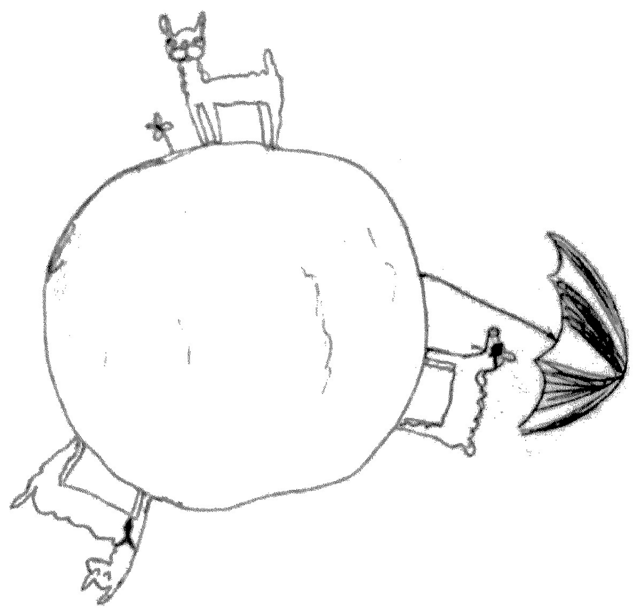
4 marks

(c) In the planet, three types of alpacas coexist in harmony: northern alpacas, tropical alpacas and southern alpacas. However, their diet is slightly different as each type eats a different type of grass. In order to avoid political struggles, the alpaca sages have decided that the surface above $z = a$ will be sown with grass for northern alpacas, the land below $z = b$ with grass for southern alpacas and the rest with grass for tropical alpacas. Furthermore, they will choose the values of a, b in such a way that the total surface of the planet corresponding to each type of alpaca is proportional with their population.

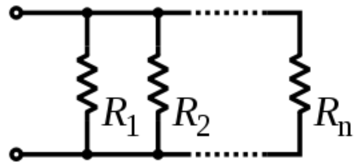
What are the values of a and b if the radius of the planet is $R = 3$[km] and the total population is composed of 10% northern alpacas, 50% tropical alpacas and 40% southern alpacas? Give your answer in [km].

Solution: From the formula above, the area of the upper hemisphere is obtained with $c = 0$. Thus the total surface of the planet is $4\pi R^2$. Imposing that $2\pi R(R - a) = 0.1 \cdot 4\pi R^2$ we obtain $a = 0.8R$. Similarly, (inverting the sphere) we obtain that $b = -0.2R$. Thus the solution is $a = 2.4[km]$ and $b = -0.6[km]$.

Answer: $a = 2.4$[km] and $b = -0.6$[km].



6. In an electrical circuit, the total resistance R of n resistors $R_1, R_2, \ldots, R_n$ connected in parallel is the reciprocal of the sum of the reciprocals of the individual resistors.



$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$$

Recall that the unit for resistance is Ohms $[\Omega]$ and that the value of a resistance is always non-negative.

5 marks

(a) You are asked to build a circuit with three resistances R_1, R_2 and R_3 connected in parallel such that $R_1 + R_2 + R_3 = 90[\Omega]$. What is the maximum total resistance $R_{\max}$ you could achieve if you are able to choose the values of R_1, R_2 and R_3?

Solution: Implicit differentiation yields $\frac{\partial R}{\partial R_i} = \frac{R^2}{R_i^2}$. Using Lagrange multipliers we obtain that there is λ such that $R^2 = \lambda R_i^2$. We deduce that $R_1 = R_2 = R_3$. Plugging this into $R_1 + R_2 + R_3 = 90[\Omega]$ we obtain that each resistance should be $30[\Omega]$ and thus $R = 10[\Omega]$.

Answer: $R_{\max} = 10[\Omega]$.

5 marks

(b) Now you are given a circuit with three resistances connected in parallel. You measure the resistances and obtain that $R_1 = \frac{1}{2}[\Omega]$,$R_2 = \frac{1}{2}[\Omega]$ and $R_3 = 1[\Omega]$ with a 1% error in each measurement. Use the method of total differentials to estimate the error in the total resistance of the circuit.

Solution: There are two possible interpretations. An error of 0.01 in each resistance or an error of $0.01R_i$. We will consider both correct. We compute $R = \frac{1}{5}[\Omega]$. Thus using total differentials we obtain that $\mathrm{d}R = \frac{R^2}{R_1^2}\mathrm{d}R_1 + \frac{R^2}{R_2^2}\mathrm{d}R_2 + \frac{R^2}{R_3^2}\mathrm{d}R_3$. Using the first interpretation we get:

$$\mathrm{d}R = \left(\frac{1}{5}\right)^2 \cdot \frac{1}{100} \cdot \left(2^2 + 2^2 + 1^2\right)$$

Thus $\mathrm{d}R = \frac{9}{2500}[\Omega]$

Using the second interpretation we get:

$$\mathrm{d}R = \left(\frac{1}{5}\right)^2 \cdot \frac{1}{100} \cdot \left(\frac{2^2}{2} + \frac{2^2}{2} + 1\right)$$

Thus $\mathrm{d}R = \frac{1}{500}[\Omega]$

Another interpretation takes 1% as relative error. The solution in that case

Answer: $\mathrm{d}R = \frac{9}{2500}[\Omega]$ or $\mathrm{d}R = \frac{1}{500}[\Omega]$.

7. Let $F(x,y,z) = x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^4 - 1$ and consider the surface implicitly defined by the equation $F(x,y,z) = 0$.

1 mark (a) Compute $\nabla F(x,y,z)$

Answer: $\nabla F(x,y,z) = (2x, y, 2z^3)$.

3 marks (b) Compute the tangent plane to the surface at $(1,1,1)$. Give it in the format $Ax + By + Cz = D$.

Solution: The tangent plane at (a,b,c) is given by the equation $\nabla F(a,b,c) \cdot (x - a, y - b, z - c) = 0$. We have that $\nabla F(1,1,1) = (2,1,2)$ and thus we obtain $2(x-1) + (y-1) + 2(z-1) = 0$, That is, $2x + y + 2z = 5$.

Answer: $2x + y + 2z = 5$.

3 marks (c) Give an equation describing the set of points on the surface at which the tangent plane is orthogonal to the xy-plane. In one word describe the curve they form.

Solution: At (a,b,c) the normal vector to the tangent plane is $\nabla(a,b,c)$. The normal vector to the xy-plane is $\vec{k} = (0,0,1)$. Thus we impose $\nabla(a,b,c) \cdot \vec{k} = 0$. This gives the condition $2a^3 = 0$ from where we deduce $z = 0$. Replacing in the equation of the surface gives $x^2 + \frac{1}{2}y^2 = 1$. An ellipse

Answer: All $(a,b,0)$ such that $a^2 + \frac{b^2}{2} = 1$. That is, an ellipse.

3 marks (d) Describe the set of points on the surface at which the tangent plane at them passes through $(0,0,0)$.

Solution: We impose that $(x,y,z) = (0,0,0)$ is a solution of $\nabla F(a,b,c) \cdot (x - a, y - b, z - c) = 0$. We obtain $-2a^2 - b^2 - 2c^4 = 0$. As $2a^2 + b^2 + c^4 = 2$ for all points in the surface we obtain that $c^4 = -2$. This equation has no solution on $\mathbb{R}$, therefore, there are no points for which the condition holds.

Answer: There are none.

8. Consider the function $f(x, y) = x^2y^2 - 3x^2y - 3xy^2 + 9xy$.

5 marks (a) Compute f_x, f_y, f_{xx}, f_{yy} and f_{xy}.

Answer: $f_x = 2xy^2 - 6xy - 3y^2 + 9y$.

Answer: $f_y = 2x^2y - 3x^2 - 6xy + 9x$.

Answer: $f_{xx} = 2y^2 - 6y$.

Answer: $f_{yy} = 2x^2 - 6x$

Answer: $f_{xy} = 4xy - 6x - 6y + 9$.

5 marks (b) Find all critical points of f. *hint:* both equations can be factorized into three terms. Try to factor either $(2x - 3)$ or $(2y - 3)$ out of them.

Solution: We impose that $\nabla F(x, y, z) = (0, 0)$. Factorizing the gradient we obtain the equations

$$y(y - 3)(2x - 3) = 0 \tag{1}$$
$$x(x - 3)(2y - 3) = 0 \tag{2}$$

Thus in (1) either $y = 0, y = 3$ or $x = \frac{1}{2}$.
If $y = 0$, then either $x = 0$ or $x = 3$ solves (2). If $y = 3$, then either $x = 0$ or $x = 3$ solves (2). If $x = \frac{3}{2}$ only $y = \frac{3}{2}$ solves (2).
We conclude that the critical points are $(0, 0), (0, 3), (3, 0), (3, 3)$ and $(\frac{3}{2}, \frac{3}{2})$.

Answer: $(0, 0), (0, 3), (3, 0), (3, 3)$ and $(\frac{3}{2}, \frac{3}{2})$.

3 marks (c) Classify the critical points of f.

Solution: We have $D(x, y) = (2x^2 - 6x)(2y^2 - 6y) - (4xy - 6x - 6y + 9)^2$. For $(0, 0), (0, 3), (3, 0)$ and $(3, 3)$ the first term is 0 and the second is nonzero, thus those are saddle points. For $(\frac{3}{2}, \frac{3}{2})$ we obtain that $D(\frac{3}{2}, \frac{3}{2}) > 0$. Also, $f_{xx}(\frac{3}{2}, \frac{3}{2}) < 0$, thus this is a local maximum.

Answer: $(0, 0), (0, 3), (3, 0), (3, 3)$ are saddle points and $(\frac{3}{2}, \frac{3}{2})$ is a local maximum

4 marks (d) Find the absolute maximum and minimum in the rectangle $D = [0, 4] \times [0, 3]$.

Solution: The only relevant critical point is $(\frac{3}{2}, \frac{3}{2})$ and $f(\frac{3}{2}, \frac{3}{2}) = \frac{81}{16}$. Along the border of D we have that f is identically zero along the lines $x = 0$, $y = 0$ and $y = 3$. Thus it remains to check the line $x = 4$ from $y = 0$ to 3. We define $g(y) = f(4, y) = 4y^2 - 12y$. Thus $g'(y) = 8y - 12y$. Plugging in this value we get $g(\frac{3}{2}) = -9$

Answer: $f_{\text{max}} = \frac{81}{16}$ at $(\frac{3}{2}, \frac{3}{2})$ and $f_{\text{min}} = -9$ at $(4, \frac{3}{2})$.

3 marks (e) Draw a sketch of a contour plot of f in the rectangle $[-1, 4] \times [-1, 4]$ with at least 5 different contours.

Solution: