

Final Examination — June 26, 2018 **Duration: 2.5 hours***This test has 8 questions on 14 pages, for a total of 100 points.*

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- Read all the questions carefully before starting to work. Unless otherwise indicated, give complete arguments and explanations for all your calculations as answers without justification will not be marked.
- Continue on a blank page if you run out of space, and **indicate this clearly on the original page.**
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	Total
Points:	20	10	10	10	10	10	10	20	100
Score:									

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
- Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - speaking or communicating with other examination candidates, unless otherwise authorized;
 - purposely exposing written papers to the view of other examination candidates or imaging devices;
 - purposely viewing the written papers of other examination candidates;
 - using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

20 marks

1. There are twelve statements below which you may classify as true or false by marking the corresponding box with an X. The rules are as follows:

- Every correct answer gives you 2 points.
- Every wrong answer gives you -1 point.
- Every answer left blank gives you 0 points.
- You may not earn less than 0 points nor more than 20 points.
- Assume that the functions f, G referenced below are smooth enough (existence of second partial derivatives, equivalence of f_{xy} and f_{yx} , well defined level curves, existence of implicit functions, etc.)

	1	2	3	4	5	6	7	8	9	10	11	12
True												
False												

1. For any two (non-zero) non-parallel vectors \vec{u}, \vec{v} in \mathbb{R}^3 , $\|\vec{u} \times \vec{v}\| > 0$.
2. If a continuous function $f(x, y)$ is positive for every (x, y) in its domain D then $\iint_D f(x, y) dA \geq 0$.
3. If a point (a, b) in the domain of a function $f(x, y)$ satisfies

$$f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)f_{yx}(a, b) = -3$$

then (a, b) is a saddle point.

4. If $\lim_{x \rightarrow 0} g(x, mx) = L$ for every value of m and $\lim_{y \rightarrow 0} g(0, y) = L$ then the limit $\lim_{(x, y) \rightarrow (0, 0)} g(x, y)$ exists and is equal to L .
5. $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if there exists $a \geq 0$ such that $\vec{u} = a\vec{v}$ or $\vec{v} = a\vec{u}$.
6. Every solution to the equations given by the method of Lagrange multipliers is a local minimum or local maximum of the associated problem.
7. For any three vectors $\vec{u}, \vec{v}, \vec{w}$ in \mathbb{R}^3 we have $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.
8. Consider the surface defined by $G(x, y, z) = 0$. If $G(a, b, c) = 0$ then any vector in the tangent plane to the surface at (a, b, c) is orthogonal to $\nabla G(a, b, c)$.
9. Let $z = f(x, y)$ and $c = f(a, b)$. The gradient $\nabla f(a, b)$ is parallel to the curve described by $f(x, y) = c$ at (a, b) .
10. Let $z = f(x, y)$. It is possible that $\nabla f(x, y) = (0, 0)$ and (x, y) is neither a minimum nor a maximum.
11. The surface described by $\frac{x^2}{a} + y^2 + z^2 = 1$ is an ellipse for all non-zero values of a .
12. If there are values a, b such that every contour of a function $z = f(x, y)$ is of the form $ax + by = c$ for some value of c then necessarily $z = f(x, y)$ describes a plane.

2. Consider the integral $I = \int_0^2 \int_{1-\frac{x}{2}}^{e^x} f(x, y) \, dy \, dx$

2 marks

(a) Sketch the region D of integration.

4 marks

(b) In the identity below the order of integration has been changed. Fill in the blanks.

$$\int_0^2 \int_{1-\frac{x}{2}}^{e^x} f(x, y) \, dy \, dx = \int_0^1 \int_{\boxed{}}^{\boxed{}} f(x, y) \, dx \, dy + \int_1^{\boxed{}} \int_{\boxed{}}^{\boxed{}} f(x, y) \, dx \, dy$$

4 marks

(c) Compute the value of I for $f(x, y) = y$.

Answer:

4 marks

3. (a) Evaluate $I = \iiint_E \sqrt{x^2 + y^2} \, dV$ over the region E bounded by $x^2 + y^2 + z^2 = 4$. *Hint:* Recall that $\sin^2(\phi) = \frac{1 - \cos(2\phi)}{2}$.

Answer:

6 marks

- (b) Consider the triple integral

$$J = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^1 f(x, y, z) \, dz dy dx.$$

Fill in the blanks below where the order of integration has been changed.

$$J = \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} f(x, y, z) \, dz dx dy$$

$$J = \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} f(x, y, z) \, dx dz dy$$

4. Consider a right circular cone of height $H > 0$ and radius $a > 0$. This cone may be defined as the solid between $z = 0$ and $z = H(1 - \frac{\sqrt{x^2+y^2}}{a})$ in the region D in the xy -plane bounded by $x^2 + y^2 = a^2$.

Suppose that the density of the cone is proportional to the distance from the z -axis. In other words, there exists a constant $k > 0$ such that $\rho(x, y) = k\sqrt{x^2 + y^2}$.

4 marks

- (a) Compute the total mass M of the cone.

Answer:

5 marks

(b) Compute the centre of mass of the cone. You may leave your answers in terms of M .

Answer:

Answer:

Answer:

1 mark

(c) Upon which of the parameters H , a and k of the problem above does the centre of mass $(\bar{x}, \bar{y}, \bar{z})$ depend?

Answer:

5. A long time ago in a galaxy far, far away, there was a little world inhabited exclusively by beautiful alpacas. Their world was so small and round that it could be modeled by a sphere $x^2 + y^2 + z^2 \leq R^2$.

2 marks

- (a) Suppose a surface is given by the equation $z = f(x, y)$ on the domain D . State the integral formula for the surface area.

Answer:

4 marks

- (b) Let $0 \leq c \leq R$ be a fixed number. Compute the total surface area S of the portion of the alpaca planet that is above $z = c$.

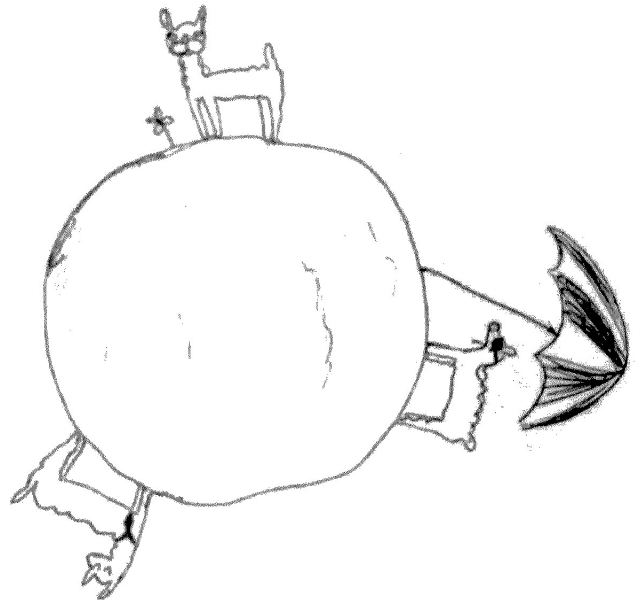
Answer:

4 marks

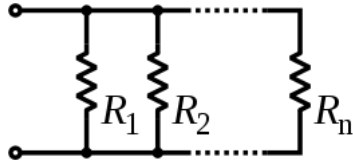
- (c) In the planet, three types of alpacas coexist in harmony: northern alpacas, tropical alpacas and southern alpacas. However, their diet is slightly different as each type eats a different type of grass. In order to avoid political struggles, the alpaca sages have decided that the surface above $z = a$ will be sown with grass for northern alpacas, the land below $z = b$ with grass for southern alpacas and the rest with grass for tropical alpacas. Furthermore, they will choose the values of a, b in such a way that the total surface of the planet corresponding to each type of alpaca is proportional with their population.

What are the values of a and b if the radius of the planet is $R = 3[\text{km}]$ and the total population is composed of 10% northern alpacas, 50% tropical alpacas and 40% southern alpacas? Give your answer in $[\text{km}]$.

Answer:



6. In an electrical circuit, the total resistance R of n resistors R_1, R_2, \dots, R_n connected in parallel is the reciprocal of the sum of the reciprocals of the individual resistors.



$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n}$$

Recall that the unit for resistance is Ohms [Ω] and that the value of a resistance is always non-negative.

5 marks

- (a) You are asked to build a circuit with three resistances R_1 , R_2 and R_3 connected in parallel such that $R_1 + R_2 + R_3 = 90[\Omega]$. What is the maximum total resistance R_{\max} you could achieve if you are able to choose the values of R_1 , R_2 and R_3 ?

Answer:

5 marks

- (b) Now you are given a circuit with three resistances connected in parallel. You measure the resistances and obtain that $R_1 = \frac{1}{2}[\Omega]$, $R_2 = \frac{1}{2}[\Omega]$ and $R_3 = 1[\Omega]$ with a 1% error in each measurement. Use the method of total differentials to estimate the error in the total resistance of the circuit.

Answer:

7. Let $F(x, y, z) = x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^4 - 1$ and consider the surface implicitly defined by the equation $F(x, y, z) = 0$.

1 mark

- (a) Compute $\nabla F(x, y, z)$

Answer:

3 marks

- (b) Compute the tangent plane to the surface at $(1, 1, 1)$. Give it in the format $Ax + By + Cz = D$.

Answer:

3 marks

- (c) Give an equation describing the set of points on the surface at which the tangent plane is orthogonal to the xy -plane. In one word describe the curve they form.

Answer:

3 marks

- (d) Describe the set of points on the surface at which the tangent plane at them passes through $(0, 0, 0)$.

Answer:

8. Consider the function $f(x, y) = x^2y^2 - 3x^2y - 3xy^2 + 9xy$.

5 marks

(a) Compute f_x , f_y , f_{xx} , f_{yy} and f_{xy} .

Answer:

Answer:

Answer:

Answer:

Answer:

5 marks

(b) Find all critical points of f . *hint*: both equations can be factorized into three terms. Try to factor either $(2x - 3)$ or $(2y - 3)$ out of them.

Answer:

3 marks

(c) Classify the critical points of f .

Answer:

4 marks

(d) Find the absolute maximum and minimum in the rectangle $D = [0, 4] \times [0, 3]$.

Answer:

3 marks

- (e) Draw a sketch of a contour plot of f in the rectangle $[-1, 4] \times [-1, 4]$ with at least 5 different contours.

This page has been left blank for your rough work and calculations.

Work on this page will *not* be marked unless you make clear indication on the original question page.