

**Midterm 2     June 13, 2018     Duration: 50 minutes**  
*This test has 4 questions on 5 pages, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, unless otherwise indicated.
- Continue on the closest blank page if you run out of space, and **indicate this clearly on the original page.**
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

### Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. Consider the surface described by the equation  $z^2(y - x^2) = x + y^2$ .

1 mark

- (a) Find  $G(x, y, z)$  such that  $(a, b, c)$  is on the surface if and only if  $G(a, b, c) = 0$ .

**Solution:** It suffices to take  $G(x, y, z) = z^2(y - x^2) - (x + y^2)$  or  $G(x, y, z) = (x + y^2) - z^2(y - x^2)$ .

Answer:  $G(x, y, z) = z^2(y - x^2) - x - y^2$ .

3 marks

- (b) Compute the gradient of  $G$  at an arbitrary point  $(a, b, c)$ .

**Solution:** Note: the negative solution is also valid.

$$\begin{aligned}\frac{\partial G}{\partial x}(a, b, c) &= -2ac^2 - 1 \\ \frac{\partial G}{\partial y}(a, b, c) &= c^2 - 2b \\ \frac{\partial G}{\partial z}(a, b, c) &= 2cb - 2ca^2\end{aligned}$$

Answer:  $\nabla G(a, b, c) = (-2ac^2 - 1, c^2 - 2b, 2cb - 2ca^2)$

4 marks

- (c) Give the equation of the tangent plane to the surface at  $(a, b, c) = (0, 1, -1)$  in the form  $Ax + By + Cz + D = 0$ .

**Solution:** The tangent plane is given by  $\nabla G(a, b, c) \cdot (x - a, y - b, z - c) = 0$ . We have  $\nabla G(a, b, c) = (-1, -1, -2)$ . Therefore we obtain

$$\begin{aligned}-(x + 0) - (y - 1) - 2(z + 1) &= 0 \\ x + y + 2z + 1 &= 0\end{aligned}$$

Answer:  $x + y + 2z + 1 = 0$ .

2 marks

- (d) Give the coordinates of a point  $P$  in the surface such that the tangent plane which passes through  $P$  is orthogonal to the  $xy$ -plane.

**Solution:** We need the normal vector  $\nabla G(a, b, c)$  to be orthogonal to the normal vector  $\vec{n} = (0, 0, 1)$  of the  $xy$ -plane. This happens exactly when  $\frac{\partial G}{\partial z}(a, b, c) = 0$ . Therefore any point on the surface satisfying  $2c(b - a^2) = 0$  will do.

Answer:  $(-1, 1, 1)$  (Other solutions possible)

4 marks

2. (a) Using the method of Lagrange multipliers, compute all the points in the surface given by the equation  $xy - z^2 + 1 = 0$  which are closest to the origin.

**Solution:** We minimize  $\|(x, y, z)\|^2$  (or  $\|(x, y, z)\|$ ) with the restriction  $xy - z^2 + 1 = 0$ . The method of Lagrange multipliers yields the conditions:

$$2x = \lambda y$$

$$2y = \lambda x$$

$$2z = -2\lambda z$$

$$xy - z^2 + 1 = 0.$$

From the first two equations we get  $2(x - y) = \lambda(y - x)$  so either  $\lambda = -2$  or  $y = x$ . If  $\lambda = -2$  we deduce  $y = -x$  and  $z = 0$ , therefore the associated solutions are  $(1, -1, 0)$  and  $(-1, 1, 0)$  both with norm  $\sqrt{2}$ . On the other hand, if  $y = x$  we deduce that either  $x = 0$  or  $\lambda = 2$ . If  $x = y = 0$  the associated solutions are  $(0, 0, 1)$  and  $(0, 0, -1)$  with norm 1. If  $\lambda = 2$  we deduce that  $z = 0$  and the remaining equation  $x^2 - 0 + 1 = 0$  has no solutions on  $\mathbb{R}$ .

Therefore the minimum distance is 1 and attained at  $(0, 0, 1)$  and  $(0, 0, -1)$ .

Answer:  $(0, 0, 1)$  and  $(0, 0, -1)$ .

2 marks

- (b) Are there any points in the surface  $xy - z^2 + 1 = 0$  which are furthest from the origin? If the answer is yes, give them, otherwise justify.

**Solution:**  $(t, \frac{t^2-1}{t}, t)$  is in the surface for arbitrarily large  $t > 0$ . Therefore the norm as a function restricted to the surface is unbounded.

Answer: There are no such points.

4 marks

- (c) A differentiable function  $z = f(x, y)$  is unknown, but an alien supercomputer gave us precise values of  $f(x, y)$  and its derivatives on points  $A, B, C$  and  $D$ .

point	$f$	$f_x$	$f_y$	$f_{xx}$	$f_{yy}$	$f_{xy}$
$A$	1	0	0	0	3	-2
$B$	1	3	0	2	2	1
$C$	2	0	0	2	3	-1
$D$	2	0	0	3	2	6

For points  $A, B, C$  and  $D$  determine whether they are a local minimum, local maximum, a saddle point, or none of the above.

**Solution:**  $A, C$  and  $D$  are critical points because  $\nabla f = 0$ , whereas  $B$  is not a critical point. In each case we compute  $f_{xx}f_{yy} - f_{xy}^2$ . In  $A$  the value is negative, thus  $A$  is a saddle point. In  $C$  it is positive and  $f_{xx}$  is positive, thus it is a local min. in  $D$  the value is negative, thus it is a saddle point.

- $A$  is: a saddle point.
- $B$  is: none of the above.
- $C$  is: a local minimum.
- $D$  is: a saddle point.

3. A bike rides on the surface given by  $f(x, y) = \sin(x^2 + y^2)$ . Seen from the sky it looks as if the bike follows on the ground the trajectory  $\vec{\gamma}(t) = (x(t), y(t)) = (3t - 5, t^2 - 3)$ .

2 marks

- (a) Compute  $\nabla f$  at an arbitrary point  $(a, b)$ .

**Solution:**  $\nabla(a, b) = (\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b))$ . A direct computation yields

$$\frac{\partial f}{\partial x} = 2x \cos(x^2 + y^2) \text{ and } \frac{\partial f}{\partial y} = 2y \cos(x^2 + y^2).$$

Answer:  $\nabla f(a, b) = (2a \cos(a^2 + b^2), 2b \cos(a^2 + b^2))$ .

2 marks

- (b) Compute the directional derivative of  $f$  at  $(1, 1)$  in the direction  $(3, 4)$ .

**Solution:** The direction vector is  $\vec{u} = (3, 4)/\|(3, 4)\|$  so  $\vec{u} = (\frac{3}{5}, \frac{4}{5})$ . We have that  $D_{\vec{u}}f(a, b) = \nabla f(a, b) \cdot \vec{u}$ . Therefore  $D_{\vec{u}}f(a, b) = \frac{2}{5} \cos(a^2 + b^2)(3a + 4b)$ .

Answer:  $D_{\vec{u}}f(a, b) = \frac{14}{5} \cos(2)$ .

2 marks

- (c) Compute  $\frac{df}{dt}$  at  $t = 2$  using the chain rule.

**Solution:** We use the chain rule  $\frac{df}{dt}(2) = \frac{\partial f}{\partial x}(x(2), y(2)) \frac{dx}{dt}(2) + \frac{\partial f}{\partial y}(x(2), y(2)) \frac{dy}{dt}(2)$ . Note that at  $t = 2$  we have  $(x(t), y(t)) = (1, 1)$ . Therefore,

$$\frac{df}{dt}(2) = 2 \cos(2) \cdot 3 + 2 \cos(2) \cdot 4$$

Answer:  $14 \cos(2)$ .

2 marks

- (d) (Rocket-powered bike). Suppose that, in the parametrization  $\vec{\gamma}(t)$  described above, the variable  $t$  represents time. What would be the speed of the bike over the surface at time  $t = 2$ ? Hint: recall that the speed of the bike is the norm of its 3D-velocity vector.

**Solution:** We have that  $\vec{\gamma}'(t) = (x'(t), y'(t), z'(t)) = (3, 2t, \frac{df}{dt}(t))$ . Therefore  $\|\vec{\gamma}'(2)\| = \|(3, 4, 14 \cos(2))\|$ .

Answer:  $\sqrt{25 + (14 \cos(2))^2}$ .

2 marks

- (e) (Monkey-powered bike) A second bike –which is being driven by a monkey– travels on the same trajectory but its speed projected to the  $xy$ -plane is constant and equal to 1. How faster is the rocket-powered bike compared to the monkey-powered bike when they pass through  $(1, 1, \sin(2))$ ?

**Solution:** The speed vector at  $(1, 1)$  is now given by  $\vec{v} = (\frac{3}{5}, \frac{4}{5})$ . We obtain that the speed is  $\|(\frac{3}{5}, \frac{4}{5}, D_{\vec{v}}f(1, 1))\| = \sqrt{1 + (\frac{14}{5} \cos(2))^2}$ . This is 5 times slower than the rocket-fueled bike.

Answer: 5 times.

- 4 marks 4. (a) Compute the integral  $I = \int_0^1 \int_{y^2}^1 y \sin(x^2) dx dy$ .

**Solution:** The antiderivative of  $\sin(x^2)$  is not elementary, therefore we change the order of integration. We obtain  $I = \int_0^1 \int_0^{\sqrt{x}} y \sin(x^2) dy dx$ . This in turn yields  $I = \int_0^1 \frac{x \sin(x^2)}{2} dx = \left( \frac{-\cos(x^2)}{4} \right)_0^1 = \frac{1-\cos(1)}{4}$

Answer:  $I = \frac{1-\cos(1)}{4}$ .

- 4 marks (b) Let  $P$  be the quarter-pizza region on  $x \geq 0$  bounded by the curves  $y = -x$ ,  $y = x$  and  $x^2 + y^2 = 1$ . Compute  $J = \int_P \sqrt{1-x^2} dA$ . Hint: you may want to try integrating first in  $y$  and thus separate the integral into two parts as in the figure below.

**Solution:** Therefore we may write  $J$  as an iterated integral as

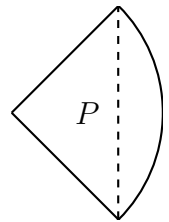
$$J = \int_0^{\frac{1}{\sqrt{2}}} \int_{-x}^x \sqrt{1-x^2} dy dx + \int_{\frac{1}{\sqrt{2}}}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2} dy dx$$

Integrating both with respect to  $y$  we get

$$J = \int_0^{\frac{1}{\sqrt{2}}} 2x\sqrt{1-x^2} dx + \int_{\frac{1}{\sqrt{2}}}^1 2(1-x^2) dx$$

The left part is computed with the change of variables  $u = 1 - x^2$  and yields  $\frac{2}{3} - \frac{1}{3\sqrt{2}}$ . The right part gives  $2 + \frac{1}{3\sqrt{2}} - \frac{2}{3} - \frac{2}{\sqrt{2}}$ . Adding everything we get  $J = 2 - \sqrt{2}$ .

Answer:  $J = 2 - \sqrt{2}$ .



- 2 marks (c) Use the answer to the previous question to compute the volume of the solid bounded by the cylinders  $x^2 + y^2 = 1$ ,  $x^2 + z^2 = 1$  and  $y^2 + z^2 = 1$ . You may leave your answer expressed as a function of  $J$  as defined in the previous question.

**Solution:** Notice that  $V = 2 \int_D \min(\sqrt{1-x^2}, \sqrt{1-y^2}) dA$  where  $D$  is the unit disk bounded by  $x^2 + y^2 = 1$ . Clearly the minimum is attained by  $\sqrt{1-x^2}$  whenever  $y \leq x$  and by  $\sqrt{1-y^2}$  when  $x \leq y$ . We may split the disk into four equal slices, and in each the function and the domain are of the form of the previous question. This yields  $V = 8J$ .

Answer:  $V = 8J = 16 - 8\sqrt{2}$