Midterm 2 June 13, 2018 Duration: 50 minutes

*This test has **4 questions** on **5 pages**, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, unless otherwise indicated.
- Continue on the closest blank page if you run out of space, and **indicate this clearly on the original page.**
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: ____________________

Student-No: ____________________ Section: ____________________

Signature: ____________________

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

1. Consider the surface described by the equation $z^2(y - x^2) = x + y^2$.

1 mark (a) Find $G(x, y, z)$ such that (a, b, c) is on the surface if and only if $G(a, b, c) = 0$.

Solution: It suffices to take $G(x, y, z) = z^2(y - x^2) - (x + y^2)$ or $G(x, y, z) = (x + y^2) - z^2(y - x^2)$.

Answer: $G(x, y, z) = z^2(y - x^2) - x - y^2$.

3 marks (b) Compute the gradient of G at an arbitrary point (a, b, c).

Solution: Note: the negative solution is also valid.

$$\frac{\partial G}{\partial x}(a, b, c) = -2ac^2 - 1$$
$$\frac{\partial G}{\partial y}(a, b, c) = c^2 - 2b$$
$$\frac{\partial G}{\partial z}(a, b, c) = 2cb - 2ca^2$$

Answer: $\nabla G(a, b, c) = (-2ac^2 - 1, c^2 - 2b, 2cb - 2ca^2)$

4 marks (c) Give the equation of the tangent plane to the surface at $(a, b, c) = (0, 1, -1)$ in the form $Ax + By + Cz + D = 0$.

Solution: The tangent plane is given by $\nabla G(a, b, c) \cdot (x - a, y - b, z - c) = 0$. We have $\nabla G(a, b, c) = (-1, -1, -2)$. Therefore we obtain

$$-(x + 0) - (y - 1) - 2(z + 1) = 0$$

$$x + y + 2z + 1 = 0$$

Answer: $x + y + 2z + 1 = 0$.

2 marks (d) Give the coordinates of a point P in the surface such that the tangent plane which passes through P is orthogonal to the xy-plane.

Solution: We need the normal vector $\nabla G(a, b, c)$ to be orthogonal to the normal vector $\vec{n} = (0, 0, 1)$ of the xy-plane. This happens exactly when $\frac{\partial G}{\partial z}(a, b, c) = 0$. Therefore any point on the surface satisfying $2c(b - a^2) = 0$ will do.

Answer: $(-1, 1, 1)$ (Other solutions possible)

4 marks

2. (a) Using the method of Lagrange multipliers, compute all the points in the surface given by the equation $xy - z^2 + 1 = 0$ which are closest to the origin.

Solution: We minimize $||(x, y, z)||^2$ (or $||(x, y, z)||$) with the restriction $xy - z^2 + 1 = 0$. The method of Lagrange multipliers yields the conditions:

$$\begin{aligned} 2x &= \lambda y \\ 2y &= \lambda x \\ 2z &= -2\lambda z \\ xy - z^2 + 1 &= 0. \end{aligned}$$

From the first two equations we get $2(x - y) = \lambda(y - x)$ so either $\lambda = -2$ or $y = x$. If $\lambda = -2$ we deduce $y = -x$ and $z = 0$, therefore the associated solutions are $(1, -1, 0)$ and $(-1, 1, 0)$ both with norm $\sqrt{2}$. On the other hand, if $y = x$ we deduce that either $x = 0$ or $\lambda = 2$. If $x = y = 0$ the associated solutions are $(0, 0, 1)$ and $(0, 0, -1)$ with norm 1. If $\lambda = 2$ we deduce that $z = 0$ and the remaining equation $x^2 - 0 + 1 = 0$ has no solutions on $\mathbb{R}$.

Therefore the minimum distance is 1 and attained at $(0, 0, 1)$ and $(0, 0, -1)$.

Answer: $(0, 0, 1)$ and $(0, 0, -1)$.

2 marks

(b) Are there any points in the surface $xy - z^2 + 1 = 0$ which are furthest from the origin? If the answer is yes, give them, otherwise justify.

Solution: $(t, \frac{t^2-1}{t}, t)$ is in the surface for arbitrarily large $t > 0$. Therefore the norm as a function restricted to the surface is unbounded.

Answer: There are no such points.

4 marks

(c) A differentiable function $z = f(x, y)$ is unknown, but an alien supercomputer gave us precise values of $f(x, y)$ and its derivatives on points A, B, C and D.

point	f	f_x	f_y	f_{xx}	f_{yy}	f_{xy}
A	1	0	0	0	3	-2
B	1	3	0	2	2	1
C	2	0	0	2	3	-1
D	2	0	0	3	2	6

For points A, B, C and D determine whether they are a local minimum, local maximum, a saddle point, or none of the above.

Solution: A, C and D are critical points because $\nabla f = 0$, whereas B is not a critical point. In each case we compute $f_{xx}f_{yy} - f_{xy}^2$. In A the value is negative, thus A is a saddle point. In C it is positive and f_{xx} is positive, thus it is a local min. in D the value is negative, thus it is a saddle point.

- A is: a saddle point.
- B is: none of the above.
- C is: a local minimum.
- D is: a saddle point.

3. A bike rides on the surface given by $f(x, y) = \sin(x^2 + y^2)$. Seen from the sky it looks as if the bike follows on the ground the trajectory $\vec{\gamma}(t) = (x(t), y(t)) = (3t - 5, t^2 - 3)$.

2 marks (a) Compute ∇f at an arbitrary point (a, b).

Solution: $\nabla(a, b) = (\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b))$. A direct computation yields

$$\frac{\partial f}{\partial x} = 2x\cos(x^2 + y^2) \text{ and } \frac{\partial f}{\partial y} = 2y\cos(x^2 + y^2).$$

Answer: $\nabla f(a, b) = (2a\cos(a^2 + b^2), 2b\cos(a^2 + b^2))$.

2 marks (b) Compute the directional derivative of f at $(1, 1)$ in the direction $(3, 4)$.

Solution: The direction vector is $\vec{u} = (3, 4)/||(3, 4)||$ so $\vec{u} = (\frac{3}{5}, \frac{4}{5})$. We have that $D_{\vec{u}}f(a, b) = \nabla f(a, b) \cdot \vec{u}$. Therefore $D_{\vec{u}}f(a, b) = \frac{2}{5}\cos(a^2 + b^2)(3a + 4b)$.

Answer: $D_{\vec{u}}f(a, b) = \frac{14}{5}\cos(2)$.

2 marks (c) Compute $\frac{\mathrm{d}f}{\mathrm{d}t}$ at $t = 2$ using the chain rule.

Solution: We use the chain rule $\frac{\mathrm{d}f}{\mathrm{d}t}(2) = \frac{\partial f}{\partial x}(x(2), y(2))\frac{\mathrm{d}x}{\mathrm{d}t}(2) + \frac{\partial f}{\partial y}(x(2), y(2))\frac{\mathrm{d}y}{\mathrm{d}t}(2)$. Note that at $t = 2$ we have $(x(t), y(t)) = (1, 1)$. Therefore,

$$\frac{\mathrm{d}f}{\mathrm{d}t}(2) = 2\cos(2) \cdot 3 + 2\cos(2) \cdot 4$$

Answer: $14\cos(2)$.

2 marks (d) (Rocket-powered bike). Suppose that, in the parametrization $\vec{\gamma}(t)$ described above, the variable t represents time. What would be the speed of the bike over the surface at time $t = 2$? Hint: recall that the speed of the bike is the norm of its 3D-velocity vector.

Solution: We have that $\vec{\gamma}'(t) = (x'(t), y'(t), z'(t)) = (3, 2t, \frac{\mathrm{d}f}{\mathrm{d}t}(t))$. Therefore $||\vec{\gamma}'(2)|| = ||(3, 4, 14\cos(2))||$.

Answer: $\sqrt{25 + (14\cos(2))^2}$.

2 marks (e) (Monkey-powered bike) A second bike –which is being driven by a monkey– travels on the same trajectory but its speed projected to the xy-plane is constant and equal to 1. How faster is the rocket-powered bike compared to the monkey-powered bike when they pass through $(1, 1, \sin(2))$?

Solution: The speed vector at $(1, 1)$ is now given by $\vec{v} = (\frac{3}{5}, \frac{4}{5})$. We obtain that the speed is $||(\frac{3}{5}, \frac{4}{5}, D_{\vec{v}}f(1, 1))|| = \sqrt{1 + (\frac{14}{5}\cos(2))^2}$. This is 5 times slower than the rocket-fueled bike.

Answer: 5 times.

4 marks 4. (a) Compute the integral $I = \int_0^1 \int_{y^2}^1 y \sin(x^2)\ \mathrm{d}x\mathrm{d}y$.

Solution: The antiderivative of $\sin(x^2)$ is not elementary, therefore we change the order of integration. We obtain $I = \int_0^1 \int_0^{\sqrt{x}} y \sin(x^2)\ \mathrm{d}y\mathrm{d}x$. This in turn yields $I = \int_0^1 \frac{x\sin(x^2)}{2}\ \mathrm{d}x = \left(\frac{-\cos(x)^2}{4}\right)_0^1 = \frac{1-\cos(1)}{4}$

Answer: $I = \frac{1-\cos(1)}{4}$.

4 marks (b) Let P be the quarter-pizza region on $x \geq 0$ bounded by the curves $y = -x$, $y = x$ and $x^2 + y^2 = 1$. Compute $J = \int\int_P \sqrt{1-x^2}\ \mathrm{d}A$. Hint: you may want to try integrating first in y and thus separate the integral into two parts as in the figure below.

Solution: Therefore we may write J as an iterated integral as

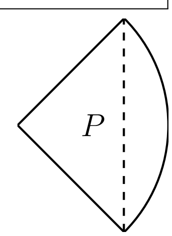
$$J = \int_0^{\frac{1}{\sqrt{2}}} \int_{-x}^{x} \sqrt{1-x^2}\ \mathrm{d}y\mathrm{d}x + \int_{\frac{1}{\sqrt{2}}}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2}\ \mathrm{d}y\mathrm{d}x$$

Integrating both with respect to y we get

$$J = \int_0^{\frac{1}{\sqrt{2}}} 2x\sqrt{1-x^2}\mathrm{d}x + \int_{\frac{1}{\sqrt{2}}}^{1} 2(1-x^2)\mathrm{d}x$$

The left part is computed with the change of variables $u = 1 - x^2$ and yields $\frac{2}{3} - \frac{1}{3\sqrt{2}}$. The right part gives $2 + \frac{1}{3\sqrt{2}} - \frac{2}{3} - \frac{2}{\sqrt{2}}$. Adding everything we get $J = 2 - \sqrt{2}$.

Answer: $J = 2 - \sqrt{2}$.



2 marks (c) Use the answer to the previous question to compute the volume of the solid bounded by the cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$. You may leave your answer expressed as a function of J as defined in the previous question.

Solution: Notice that $V = 2\int\int_D \min(\sqrt{1-x^2}, \sqrt{1-y^2})\ \mathrm{d}A$ where D is the unit disk bounded by $x^2 + y^2 = 1$. Clearly the minimum is attained by $\sqrt{1-x^2}$ whenever $y \leq x$ and by $\sqrt{1-y^2)}$ when $x \leq y$. We may split the disk into four equal slices, and in each the function and the domain are of the form of the previous question. This yields $V = 8J$.

Answer: $V = 8J = 16 - 8\sqrt{2}$