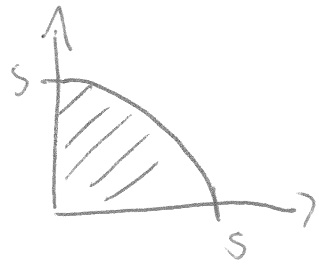
HW5 Solutions

§ 13.4 - 16



$$M = \int_0^{\pi/2} \int_0^5 (R+1)R\,dR\,d\theta$$

$$= \frac{\pi}{2}\left(\frac{R^3}{3} + \frac{R^2}{2}\right)\Bigg|_0^5 = \frac{\pi}{2}\left(\frac{125}{3} + \frac{25}{2}\right)$$

$$= \frac{\pi}{2}\left(\frac{250+75}{6}\right) = \boxed{\frac{325\pi}{12}} \text{ kg.}$$

§ 13.4 - 24

$$\bar{x} = \frac{1}{M}\int_0^{\pi/2}\int_0^5 (R\cos\theta)(R+1)R\,dR\,d\theta$$

$$\bar{x} = \frac{1}{M}\int_0^{\pi/2}\int_0^5 \cos\theta(R^3+R^2)\,dR\,d\theta = \frac{1}{M}\int_0^{\pi/2}\cos\theta\int_0^5(R^3+R^2)\,dR\,d\theta$$

$$= \frac{1}{M}\int_0^{\pi/2}\cos\theta\left(\frac{R^4}{4}+\frac{R^3}{3}\right)\Bigg|_0^5 d\theta = \frac{1}{M}\int_0^{\pi/2}\cos\theta\left(\frac{625}{4}+\frac{125}{3}\right)d\theta$$

$$= \frac{1}{M}\cdot\left(\frac{1875+500}{12}\right)\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{M}\cdot\frac{2375}{12}(\sin\theta)\Big|_0^{\pi/2}$$

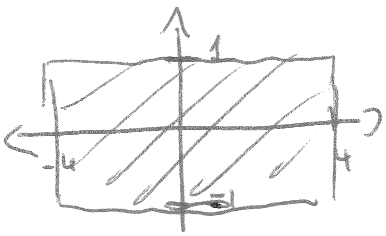
$$= \frac{12}{325\pi}\cdot\frac{2375}{12}\cdot 1 = \frac{25\cdot 95}{25\cdot 13}\cdot\frac{1}{\pi} = \boxed{\frac{95}{13\pi}}$$

By SYMMETRY $\boxed{\bar{y} = \frac{95}{13\pi}}$ so $\cancel{(\bar{x},\bar{y}) = \frac{95}{13\pi}}$

$$\boxed{(\bar{x},\bar{y}) = \left(\frac{95}{13\pi}, \frac{95}{13\pi}\right)}$$

§13.4 – 28

$$I_x = \int_{-4}^{4}\int_{-1}^{1} y^2 \cdot 1 \cdot dy\,dx$$



$$= \int_{-4}^{4} \frac{y^3}{3}\Big|_{-1}^{1} dx = \boxed{\frac{16}{3}}$$

$\delta(x,y) = 1$

$$I_y = \int_{-4}^{4}\int_{-1}^{1} x^2\,dy\,dx = 2\int_{-4}^{4} x^2\,dx = 2\,\frac{x^3}{3}\Big|_{-4}^{4} = \frac{4\cdot 64}{3} = \boxed{\frac{256}{3}}$$

so $\boxed{I_0 = I_x + I_y = \frac{272}{3}}$

§13.5 – 8

$$I = \iint_D \sqrt{1 + \|\nabla f(x,y)\|^2}\,dA.$$

$D = \{(x,y) \mid x^2+y^2 \le 9\}$

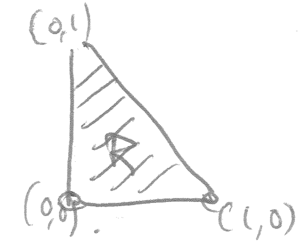
$f(x,y) = 1/(x^2+y^2+1)$ so $\nabla f(x,y) = \left(\frac{-1}{(x^2+y^2+1)^2}\cdot 2x\ ,\ \frac{-2y}{(x^2+y^2+1)^2}\right)$

so $\|\nabla f(x,y)\|^2 = \dfrac{4(x^2+y^2)}{(x^2+y^2+1)^4}$

$$\text{so } I = \iint_D \sqrt{\frac{(x^2+y^2+1)^4 + 4(x^2+y^2)}{(x^2+y^2+1)^4}}\,dA = \boxed{\int_0^{2\pi}\int_0^3 \frac{\sqrt{(R^2+1)^4 + 4R^2}}{(R^2+1)^2}\cdot R\,dR\,d\theta}$$

Also Valid (But harder) in rectangular coordinates

§ 13.5 − 12 | $f(x,y) = 2(x+y+1)$.



$$S = \iint_R \sqrt{1+\|\nabla f(x,y)\|^2}\, dA \qquad \nabla f(x,y) = (2,2).$$

$$S = \iint_R \sqrt{1+2^2+2^2}\, dA = 3\iint_R dA = 9\ \text{area}(R) = \boxed{\frac{3}{2}}$$

§ 13.6 − 10 |

$$I = \int_0^1 \int_0^2 \int_1^{3-2z} 1\, dx\, dy\, dz$$

evaluating we get $I = \int_0^1 \int_0^2 2-2z\, dy\, dz = 4\int_0^1 1-z\, dz$

$$= 4\left(z - \frac{z^2}{2}\right)\Big|_0^1 = \frac{4}{2} = \boxed{2}$$

§ 13.6 − 22 | ,

$$\bar{x} = \frac{1}{2}\int_0^1\int_0^2\int_1^{3-2z} x\, dx\, dy\, dz = \int_0^1 \left(\frac{x^2}{2}\right)\Big|_1^{3-2z} dz = \int_0^1 \frac{9-12z+4z^2-1}{2}\, dz$$

$$= \int_0^1 4-6z+2z^2\, dz = 4z-3z^2+\frac{2z^3}{3}\Big|_0^1 = 4-3+\frac{2}{3} = \boxed{\frac{5}{3}}$$

$$\bar{y} = \frac{1}{2}\int_0^1\int_0^2\int_1^{3-2z} y\, dx\, dy\, dz = \frac{1}{2}\int_0^1 2-2z \int_0^2 y\, dy\, dz = \left(z-\frac{z^2}{2}\right)\Big|_0^1 \cdot \left(\frac{y^2}{2}\right)\Big|_0^2 = \boxed{1}$$

$$\bar{z} = \frac{1}{2}\int_0^1\int_0^2\int_1^{3-2z} z\, dx\, dy\, dz = \int_0^1 z(2-2z)\, dz = \boxed{\frac{1}{3}}$$

$2\left(\frac{1}{2}-\frac{1}{3}\right)$

so $\boxed{(\bar{x},\bar{y},\bar{z}) = \left(\frac{5}{3}, 1, \frac{1}{3}\right)}$

EARTH AV DIST

in Spherical North pole is $\phi = 0$, $\rho = R$, $\theta =$ whatever (say 0).

a point on earth is given by $(\theta, \phi, \rho = R)$.

distance is $\|(0,0,R) - (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)\|$

$$= \sqrt{R^2\cos^2\theta\sin^2\phi + R^2\sin^2\phi\sin^2\theta + R^2(1-\cos\phi)^2}$$

$$= R\sqrt{\sin^2\phi + 1 - 2\cos\phi + \cos^2\phi}$$

$$= R\sqrt{2-2\cos\phi} = \sqrt{2}R\sqrt{1-\cos\phi}$$

So ~~δ~~ Av ~~distance~~ is

$$\delta_{AV} = \frac{1}{4\pi R^2}\int_0^{2\pi}\int_0^{\pi}\sqrt{2}R\sqrt{1-\cos\phi}\,R^2\sin\phi\,d\phi\,d\theta$$

$$= \frac{\sqrt{2}R^3}{4\pi R^2}\int_0^{2\pi}\int_0^{\pi}\sin\phi\sqrt{1-\cos\phi}\,d\phi\,d\theta$$

$$= \frac{\sqrt{2}}{2}R\int_0^{\pi}\sin\phi\sqrt{1-\cos\phi}\,d\phi \qquad \begin{aligned} u &= 1-\cos\phi \\ du &= \sin\phi\,d\phi \end{aligned}$$

$$= \frac{R}{\sqrt{2}}\int_0^{2}\sqrt{u}\,du = \frac{R}{\sqrt{2}}\,\frac{2}{3}u^{3/2}\Big|_0^2 = \frac{2R}{3\sqrt{2}}\cdot 2\sqrt{2} = \boxed{\frac{4R}{3}}$$

$$\boxed{\delta_{AV} = 4R/3}$$