Solutions HW 3

§ 12.5 8 | $z = x^2 - y^2$ $x(t) = t$ $y(t) = t^2 - 1$ $t = 1$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2x \cdot 1 - 2y \cdot 2t.$$

~~$2t - 2t^2 \cdot 2t = 2t(1 - 2t^2)$~~

$$= 2t - 2(t^2 - 1) \cdot 2t = 2t(1 - 2(t^2 - 1))$$

$$= 2t(3 - 2t^2)$$

$$= \boxed{-4t^3 + 6t.}$$

So $\frac{dz}{dt}(1) = -4(1)^3 + 6(1) = \boxed{2}$

§ 12.5 14 | $-4t^3 + 6t = 0$

either $t = 0$ or $6 - 4t^2 = 0$

so $\boxed{t = 0, \quad t = \frac{\sqrt{6}}{2} \quad t = -\frac{\sqrt{6}}{2}.}$

§ 12.5 20 | $z = \cos(\pi x + \frac{\pi}{2} y)$ $x = st^2$ $y = s^2 t$. $s = 1, t = 1$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = -\pi \sin(\pi x + \tfrac{\pi}{2} y) t^2 - \frac{\pi}{2} \sin(\pi x + \tfrac{\pi}{2} y)(2st)$$

$$= -(t + s)\left(t\pi \sin\left(\pi x + \tfrac{\pi}{2} y\right)\right)$$

So $\frac{\partial z}{\partial s}(1, 1) = -2\left(\pi \sin\left(\pi + \frac{\pi}{2}\right)\right) = 2\pi.$

$$\frac{\partial z}{\partial t} = -\pi \sin(\pi x + \tfrac{\pi}{2} y) \cdot 2st - \frac{\pi}{2} \sin(\pi x + \tfrac{\pi}{2} y) \cdot s^2$$

$$= -\left(\pi s \sin(\pi x + \tfrac{\pi}{2} y)\right)\left(2t + \frac{s}{2}\right)$$

So $\frac{\partial z}{\partial t}(1, 1) = \pi \cdot (2 + \frac{1}{2}) = \frac{5\pi}{2}.$

§ 12.5 24 $(3x^2+2y^3)^4 = 2.$

$F(x,y) = 0 \qquad F(x,y) = (3x^2+2y^3)^4 - 2.$

$F_x = 4(3x^2+2y^3)^3 \cdot 6x$

$F_y = 4(3x^2+2y^3)^3 \cdot 6y^2$

Suppose $y = g(x)$.

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x} = 0$$

$$\Rightarrow \frac{\partial y}{\partial x} = -F_x/F_y.$$

so $\boxed{\frac{\partial y}{\partial x} = \frac{-x}{y^2}}$

§ 12.6 8 $f(x,y) = \sin x \cos y.$

$\nabla f(x,y) = (\cos x \cos y, -\sin x \sin y).$

§ 12.6 14 $P = \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

$$\nabla f\left(\tfrac{\pi}{4}, \tfrac{\pi}{3}\right) = \left(\frac{\sqrt2}{2}\cdot\frac12, -\frac{\sqrt2}{2}\cdot\frac{\sqrt3}{2}\right) = \left(\frac{\sqrt2}{4}, -\frac{\sqrt6}{4}\right).$$

(a) $$\nabla f\left(\tfrac{\pi}{4}, \tfrac{\pi}{3}\right)\cdot\frac{(1,1)}{\|(1,1)\|} = \frac{\left(\frac{\sqrt2}{4}, -\frac{\sqrt6}{4}\right)\cdot(1,1)}{\sqrt2}$$

$$= \frac14 - \frac{\sqrt3}{4} = \boxed{\frac{1-\sqrt3}{4}}$$

(b) $$\nabla f\left(\tfrac{\pi}{4}, \tfrac{\pi}{3}\right)\cdot\frac{\left(-\frac{\pi}{4}, -\frac{\pi}{3}\right)}{\left\|\left(-\frac{\pi}{4}, -\frac{\pi}{3}\right)\right\|} = \frac{1}{\sqrt{\frac{\pi^2}{16}+\frac{\pi^2}{9}}}\cdot\left(-\frac{\sqrt2}{4}\cdot\frac{\pi}{4} + \frac{\sqrt6}{4}\cdot\frac{\pi}{3}\right)$$

$$= \frac{12}{5}\left(-\frac{\sqrt2}{16} + \frac{\sqrt6}{12}\right) = \boxed{\frac{\sqrt6}{5} - \frac{3\sqrt2}{20}}$$

§ 12.8 | 12

$f(x,y) = x^4 - 2x^2 + y^3 - 27y - 15$

$\nabla f(x,y) = (4x^3 - 4x \, , \, 3y^2 - 27)$.

so $(y = 3$ or $y = -3)$ and $(x = 0$ or $x = 1$ or $x = -1)$.

6 critical points to check!.

$(0,3), (-1,3), (1,3), (0,-3), (\phi,-3), (-1,-3)$

$f_{xx} = 12x^2 - 4 \qquad f_{yy} = 6y \qquad f_{xy} = 0$

So $H(x,y) = (12x^2 - 4) \cdot 6y$.

CP $(-1,-3) \mapsto H(-1,-3) = 8 \cdot 6 \cdot -3 = -144$ SADDLE.

CP $(-1,3) \mapsto H(-1,3) = 8 \cdot 6 \cdot 3 = 144$
as $f_{xx} > 0$ LOCAL MIN.

CP $(0,-3) \mapsto H(0,-3) = -4 \cdot 6 \cdot -3 = 72$.
$f_{xx} < 0$ LOCAL MAX.

CP $(0,3) \mapsto H(0,3) = -4 \cdot 6 \cdot 3 = -72$ SADDLE.

CP $(1,-3) \mapsto H(1,-3) = -144$ SADDLE.
CP $(1,3) \mapsto H(1,3) = 144$
$f_{xx} > 0$ Local MIN

SAME AS $(-1,-3)$ $(-1,3)$

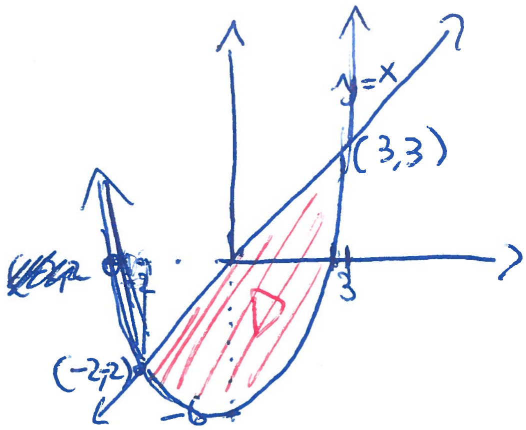
extra: $f(x,y) = 3(y+1)^2 - 2x^2$

1) Domain $y = x$ $\quad y = x^2 - 6$.

$x = x^2 - 6$

$x^2 - x - 6 = 0$.

$x = \frac{1 \pm \sqrt{1+24}}{2} = \begin{cases} 3 \\ -2 \end{cases}$



2) $\nabla f(x,y) = (-4x, 6(y+1))$

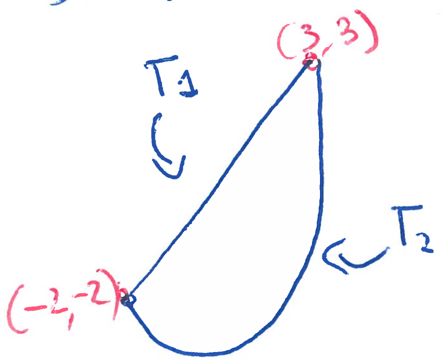
so the critical points ~~are~~ is $(0,-1)$.

$f_{xx} = -4 \quad f_{yy} = 6 \quad f_{xy} = 0$.

so $H(x,y) = -24 \quad \therefore H(0,-1) < 0$

$\Rightarrow (0,-1)$ is a SADDLE

3) We need to work on the Boundary.



$T_1(t) = (5t-2, 5t-2)$ for $t \in [0,1]$.

$T_2(t) = (5t-2, (5t-2)^2 - 6)$

$= (5t-2, 25t^2 - 20t - 2)$ for $t \in [0,1]$

So $f(T_1(t)) = 3(5t-2+1)^2 - 2(5t-2)^2$

$= 3(5t-1)^2 - 2(5t-2)^2$

~~Extreme pts (-2,-2) and (3,3) give.~~

$f(T_2(t)) = 3(25t^2 - 20t - 1)^2 - 2(5t-2)^2$.

NOW WE CHECK $(-2,-2)$, $(3,3)$ and Γ_1, Γ_2.

$f(-2,-2) = -5$

$f(3,3) = 30$.

in Γ_1. $\frac{df}{dt}(t) = 5\cdot 6(5t-1) - 5\cdot 4(5t-2)$.

$= 5(30t - 6 - 20t + 8)$

$= 5(10t + 2)$

$= 10(5t+1)$

so $\frac{df}{dt}(t) = 0$ gives $t = -\frac{1}{5}$.

ie $x = -3$ (outside of DOMAIN)

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in Γ_2 $\frac{df}{dt}(t) = 6(25t^2 - 20t - 1)(50t - 20) - 20(5t-2)$

$= (5t-2)\left(60(25t^2-20t-1) - 20\right)$

if $t = +\frac{2}{5}$ ~~(outside of domain)~~ ~~X~~ $\Rightarrow (x,y) = (0,$ [illegible]

$\Downarrow$

$f(x,y) = 75$

so $60(25t^2 - 20t - 1) = 20$

ie: $25t^2 - 20t = \frac{4}{3}$.

so $t = \dfrac{20 \pm \sqrt{400 + \frac{400}{3}}}{50}$

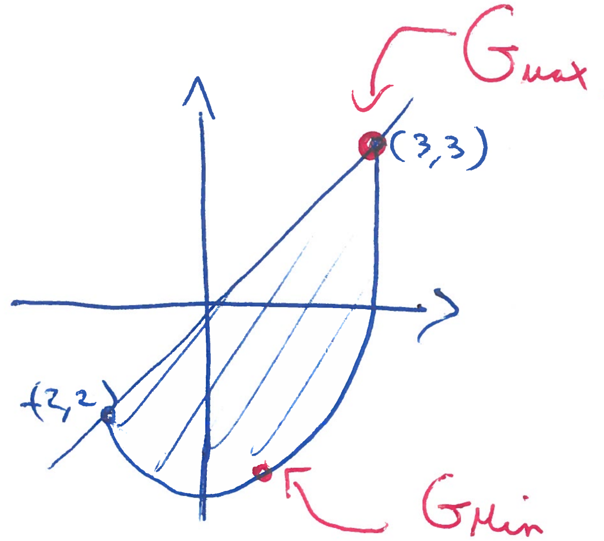
$= \frac{20}{50}\left(1 \pm \frac{2}{\sqrt{3}}\right)$

$t = \frac{2}{5}\left(1 - \frac{2}{\sqrt{3}}\right) < 0$

so out of domain

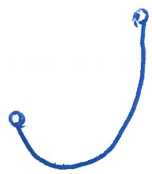
eval $t = \frac{2}{5}\left(1 + \frac{2}{\sqrt{3}}\right) \mapsto x = \frac{4}{\sqrt{3}}$, $y = \frac{-2}{3}$ so $f(x,y) = -\frac{31}{3}$.

Let's recap.



in the domain there is only a saddle point (0,-1) so no min nor Max.

in Γ_1 =  the critical point occurs outside the bounds.

in Γ_2 =  there is a critical point at $\left(\frac{4}{\sqrt{3}}, \frac{-2}{3}\right)$ with value $-\frac{31}{3}$. and one at (0,6) with value 75

$f(-2,-2) = -5$

$f(3,3) = 30$

So

- global min at $\left(\frac{4}{\sqrt{3}}, \frac{-2}{3}\right)$ with value $-31/3$
- global Max at $(0,-6)$ with value 75