

**Final Examination — December 16, 2015**      **Duration: 2.5 hours***This test has 10 questions on 12 pages, for a total of 80 points.*

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- Read all the questions carefully before starting to work. Give complete arguments and explanations for all your calculations. With the exception of #4, answers without justification will not be marked.
- Continue on the back of the *previous page* if you run out of space, *with clear indication on the original page* that your solution is continued elsewhere.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	7	8	9	9	7	10	10	6	7	7	80
Score:											

**Student Conduct during Examinations**

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. Consider the unit sphere consisting of points  $(x, y, z)$  with  $x^2 + y^2 + z^2 = 1$ ,

1 mark

(a) Determine a normal vector to the tangent plane at a point  $(x, y, z)$  on the sphere.

Answer:  $\langle 2x, 2y, 2z \rangle$

**Solution:** Let  $F(x, y, z) = x^2 + y^2 + z^2$ .  
A normal vector is given by  $\vec{\nabla}F = \langle 2x, 2y, 2z \rangle$ .

2 marks

(b) Determine the equation of the tangent plane at the point  $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ , and determine the equation of the tangent plane at the point  $(\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$ .

Answer:  $x + \sqrt{3}y = 2$  and  
 $x + \sqrt{3}z = 2$ , respectively

**Solution:** Using the normal from part (a), the equations are

$$x + \sqrt{3}y = \frac{1}{2} + \sqrt{3} \frac{\sqrt{3}}{2} = 2$$

and

$$x + \sqrt{3}z = \frac{1}{2} + \sqrt{3} \frac{\sqrt{3}}{2} = 2.$$

2 marks

(c) Determine the cosine of the acute angle  $\theta$  between the two planes in part (b).

Answer:  $\cos \theta = \frac{1}{4}$

**Solution:**

$$\cos \theta = \frac{\langle 1, \sqrt{3}, 0 \rangle \cdot \langle 1, 0, \sqrt{3} \rangle}{2 \cdot 2} = \frac{1}{4}.$$

2 marks

(d) Determine the equation of the line of intersection of the two planes in part (b), in symmetric form.

Answer:  $\frac{x-2}{-\sqrt{3}} = y = z$

**Solution:** Comparing the two equations, we see that  $y = z$ . The first equation also gives  $y = \frac{x-2}{-\sqrt{3}}$ .

2. Wheat production  $W$  in a given year depends on the average temperature  $T$  and the rainfall  $R$ . It is estimated that, at current production levels,  $\frac{\partial W}{\partial T} = -2$  Kt/ $^{\circ}\text{C}$  (kilotonnes per Centigrade degree) and  $\frac{\partial W}{\partial R} = 8$  Kt/cm (kilotonnes per cm).

4 marks

- (a) It is estimated that the average temperature is rising at a rate of  $0.15^{\circ}\text{C}/\text{year}$  and rainfall is decreasing at a rate of  $0.1$  cm/year. Using this estimated data, what is the current rate of change  $\frac{dW}{dt}$  of wheat production (give the units too).

Answer:  $-1.1$  Kt/year

**Solution:** By the chain rule,

$$\begin{aligned}\frac{dW}{dt} &= \frac{\partial W}{\partial T} \frac{dT}{dt} + \frac{\partial W}{\partial R} \frac{dR}{dt} \\ &= (-2)(0.15) + (8)(-0.1) = -0.3 - 0.8 = -1.1.\end{aligned}$$

4 marks

- (b) Suppose that the rainfall this year actually decreased by  $0.08$  cm while the average temperature increased by  $0.2^{\circ}\text{C}$ . Using differentials, estimate the actual change in production this year (give the units).

Answer:  $-1.04$  Kt

**Solution:**

$$\begin{aligned}dW &= \frac{\partial W}{\partial T} dT + \frac{\partial W}{\partial R} dR \\ &= (-2)(0.2) + (8)(-0.08) = -0.4 - 0.64 = -1.04.\end{aligned}$$

3. Consider the function  $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$  on the disk  $D$  given by  $x^2 + y^2 \leq 4$ . You may use the fact that  $e \approx 2.71828$ .

5 marks

- (a) Determine the critical points of  $f$  inside  $D$  and the values of  $f$  at those critical points.

$$\text{Answer: } f(0, 0) = 0, \\ f(0, \pm 1) = 2e^{-1}, f(\pm 1, 0) = e^{-1}.$$

**Solution:** The partial derivatives are:

$$f_x = e^{-x^2-y^2}(-2x(x^2 + 2y^2) + 2x) = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2), \\ f_y = e^{-x^2-y^2}(-2y(x^2 + 2y^2) + 4y) = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2).$$

For  $f_x = 0$  we need  $x = 0$  or  $x^2 + 2y^2 = 1$ , and for  $f_y = 0$  we need  $y = 0$  or  $x^2 + 2y^2 = 2$ . The critical points are therefore:

$$(0, 0), (0, 1), (0, -1), (1, 0), (-1, 0).$$

At these points, we have  $f(0, 0) = 0$ ,  $f(0, \pm 1) = 2e^{-1}$ ,  $f(\pm 1, 0) = e^{-1}$ .

3 marks

- (b) Determine the absolute maximum and absolute minimum values of  $f$  on the boundary of  $D$ .

$$\text{Answer: maximum } f(0, \pm 2) = 8e^{-4}, \text{ minimum } f(\pm 2, 0) = 4e^{-4}.$$

**Solution:** On the boundary,  $f(\pm\sqrt{4-y^2}, y)$  takes values  $e^{-4}(4+y^2)$  with  $-2 \leq y \leq 2$ . This has maximum  $f(0, \pm 2) = 8e^{-4}$  and minimum  $f(\pm 2, 0) = 4e^{-4}$ .

Or:

Use Lagrange multipliers with  $g(x) = x^2 + y^2 = 4$  and solve:

$$\lambda 2x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2), \\ \lambda 2y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2).$$

Cancelling  $2x$  in first equation and  $2y$  in second leads to inconsistent equations. We get a solution from  $x = 0$  and  $0^2 + y^2 = 4$ , namely  $(0, \pm 2)$ . We get a solution also from  $y = 0$  and  $x^2 + 0^2 = 4$ , namely  $(\pm 2, 0)$ .

So again we get maximum  $f(0, \pm 2) = 8e^{-4}$  and minimum  $f(\pm 2, 0) = 4e^{-4}$ .

1 mark

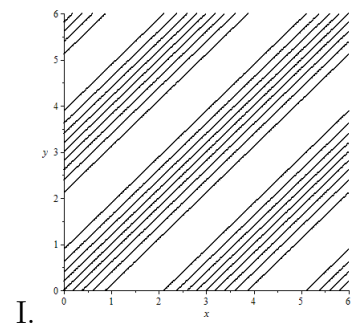
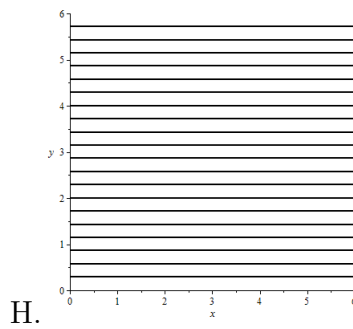
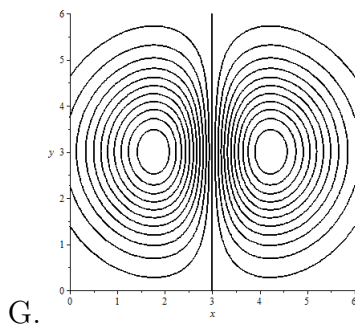
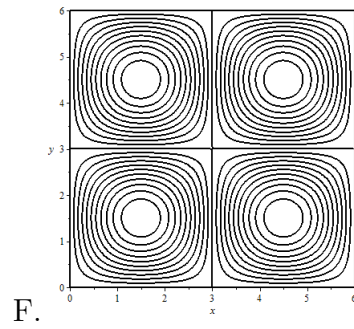
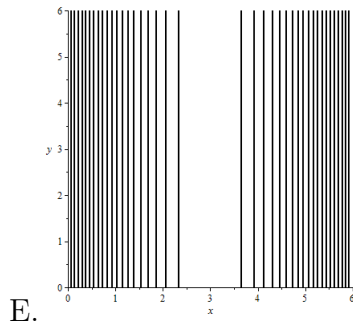
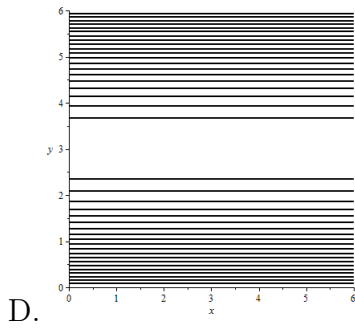
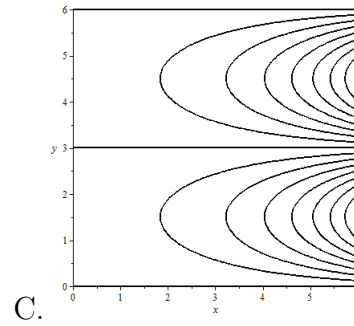
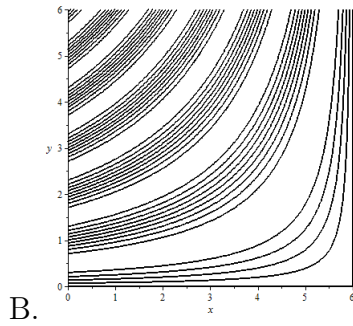
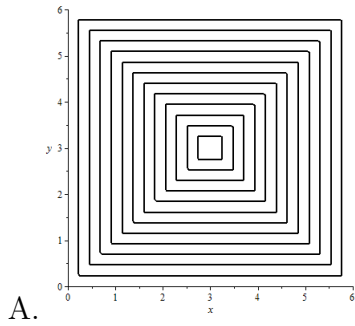
- (c) What are the locations and values of the absolute maximum and absolute minimum of  $f$  on  $D$ ?

$$\text{Answer: minimum } f(0, 0) = 0, \\ \text{maximum } f(0, \pm 1) = 2e^{-1}.$$

**Solution:** We seek the smallest and largest values among:  $0, 4e^{-4}, 8e^{-4}, e^{-1}, 2e^{-1}$ . They are listed here in order.

9 marks

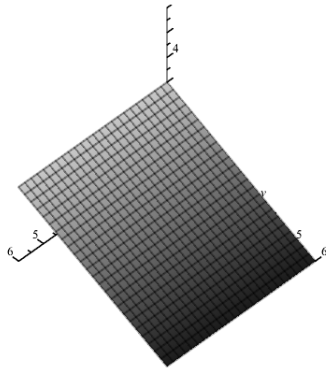
4. Consider the following 9 contour plots and 9 graphs (next page). Each contour plot is the contour plot of one of the 9 graphs. Match each contour plot with the corresponding graph. In the 9 contour plots, the  $x$  axis is horizontal, the  $y$  axis is vertical and the *values* of the contours are evenly spaced. Write the number corresponding to the matching graph next to the letter labelling each contour plot.



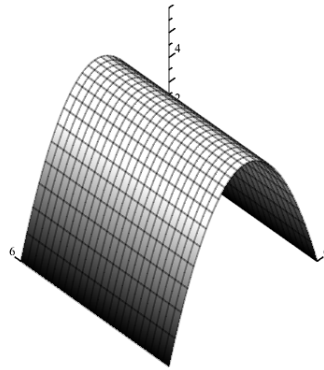
**Solution:** A4, B9, C7, D3, E2, F5, G8, H1, I6

In the 9 graphs below, the positive  $x$  axis is on the left, the positive  $y$  axis is on the right, and the positive  $z$  axis is upward.

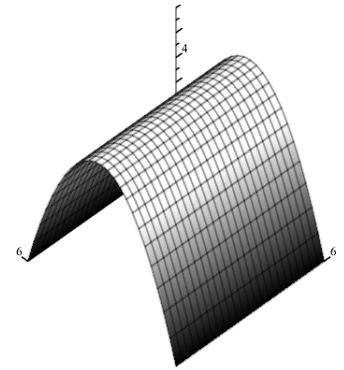
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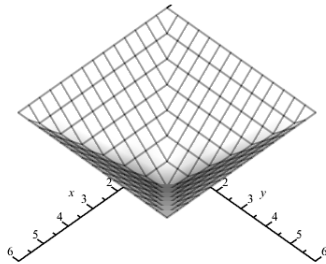
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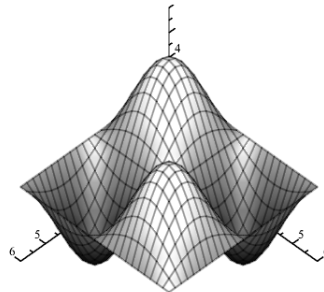
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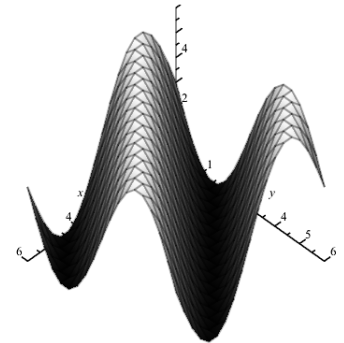
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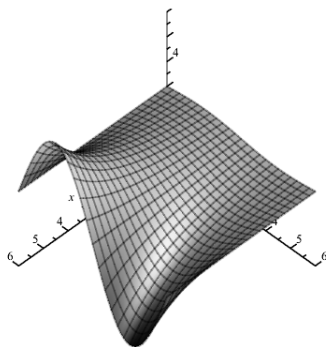
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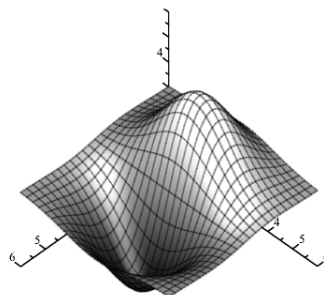
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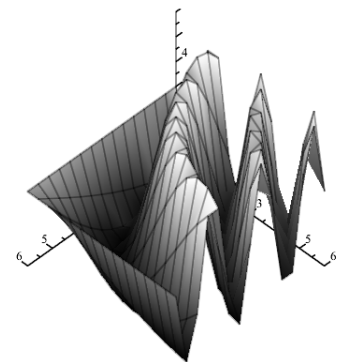
6.



7.



8.



9.

5. Consider a hill whose height is described by  $f(x, y) = 100 - \frac{1}{2}x^2 - \frac{1}{2}y^2$ , measured in metres.

3 marks

- (a) At time  $t = 0$ , I start walking on the hill at position  $(5, 5, 75)$ . I am walking at 1 metre/sec, and I set out in the direction  $\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$ . At what rate is my altitude changing at time 0?

Answer:  $-\frac{5}{2}(1 + \sqrt{3})$  m/s

**Solution:** First,  $\vec{\nabla}f = \langle -x, -y \rangle$ , so  $\vec{\nabla}f(5, 5) = \langle -5, -5 \rangle$ . Let  $\vec{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$ . My initial rate of change is the directional derivative

$$D_{\vec{u}}f(5, 5) = \vec{\nabla}f(5, 5) \cdot \vec{u} = \langle -5, -5 \rangle \cdot \langle 1, \sqrt{3} \rangle \frac{1}{2} = -\frac{5}{2} (1 + \sqrt{3}).$$

1 mark

- (b) Find an upward-pointing normal vector  $\vec{n}$  to the surface of the hill.

Answer:  $\vec{n} = \langle x, y, 1 \rangle$

**Solution:** Let  $F(x, y, z) = z - f(x, y) = z + \frac{1}{2}x^2 + \frac{1}{2}y^2 - 100$ .

An upward-pointing normal is  $\vec{\nabla}F = \langle x, y, 1 \rangle$ .

3 marks

- (c) Douglas fir trees grow vertically (in the  $z$ -direction) on the surface of the hill. Where on the hill is the angle  $\alpha$  between the tree trunks and the normal vector given by  $\alpha = \frac{\pi}{3}$ ?

Answer: On the hill directly above the circle of radius  $\sqrt{3}$  centred at the origin of the  $x$ - $y$  plane.

**Solution:** The trees point upwards in the  $\hat{k}$  direction. We want

$$\cos \alpha = \frac{1}{2} = \frac{\vec{\nabla}F \cdot \hat{k}}{|\vec{\nabla}F| |\hat{k}|} = \frac{\langle x, y, 1 \rangle \cdot \langle 0, 0, 1 \rangle}{|\vec{\nabla}F|} = \frac{1}{|\vec{\nabla}F|} = \frac{1}{\sqrt{x^2 + y^2 + 1}}.$$

That is

$$\begin{aligned} \sqrt{x^2 + y^2 + 1} &= 2 \\ x^2 + y^2 + 1 &= 4 \\ x^2 + y^2 &= 3 \end{aligned}$$

6. Consider the integral

$$\int_0^2 \int_{y^2}^4 y^3 e^{x^3} dx dy.$$

5 marks

(a) Write the integral in reversed order.

$$\text{Answer: } \int_0^4 \int_0^{\sqrt{x}} y^3 e^{x^3} dy dx$$

**Solution:**

$$\int_0^2 \int_{y^2}^4 y^3 e^{x^3} dx dy = \int_0^4 \int_0^{\sqrt{x}} y^3 e^{x^3} dy dx.$$

5 marks

(b) Using the result of part (a), evaluate the integral.

$$\text{Answer: } \frac{1}{12}(e^{64} - 1)$$

**Solution:**

$$\begin{aligned} \int_0^4 \int_0^{\sqrt{x}} y^3 e^{x^3} dy dx &= \int_0^4 \left[ \frac{y^4}{4} e^{x^3} \right]_{y=0}^{y=\sqrt{x}} dx \\ &= \int_0^4 \frac{x^2}{4} e^{x^3} dx. \end{aligned}$$

Then we make the substitution  $u = x^3$  to evaluate

$$\int_0^4 \frac{x^2}{4} e^{x^3} dx = \int_0^{64} \frac{1}{12} e^u du = \frac{1}{12} [e^u]_{u=0}^{u=64} = \frac{e^{64} - 1}{12}.$$



7. Consider the wedge-shaped region contained inside the cylinder  $x^2 + y^2 = 9$ , bounded above by the plane  $z = x$ , and bounded below by the  $xy$  plane.

5 marks

- (a) Write a double integral (including limits of integration) whose value is the volume of the wedge-shaped region.

Answer:

$$V = \int_{-\pi/2}^{\pi/2} \int_0^3 r \cos \theta r dr d\theta$$

**Solution:**

$$V = \int_{-\pi/2}^{\pi/2} \int_0^3 r \cos \theta r dr d\theta.$$

Or:

$$V = \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x dy dx.$$

5 marks

- (b) Evaluate the integral in part (a) to determine the volume of the wedge-shaped region.

Answer: 18

**Solution:**

$$\begin{aligned} V &= \int_{-\pi/2}^{\pi/2} \int_0^3 r \cos \theta r dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \cos \theta \frac{1}{3} r^3 \Big|_0^3 d\theta \\ &= 9 \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = 9 \sin \theta \Big|_{-\pi/2}^{\pi/2} = 18. \end{aligned}$$

Or:

$$\begin{aligned} V &= \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x dy dx \\ &= \int_0^3 x 2\sqrt{9-x^2} dx \\ &= - \int_9^0 u^{1/2} du \quad (u = 9 - x^2) \\ &= \frac{2}{3} u^{3/2} \Big|_0^9 = 18. \end{aligned}$$

6 marks

8. Determine the surface area of the surface given by  $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ , over the square  $0 \leq x \leq 1, 0 \leq y \leq 1$ .

Answer:  $\frac{4}{15}(3^{5/2} - 1)$

**Solution:** Since  $\frac{\partial z}{\partial x} = x^{1/2}$  and  $\frac{\partial z}{\partial y} = y^{1/2}$ ,

$$\begin{aligned} A &= \int_0^1 \int_0^1 \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy \\ &= \int_0^1 \int_0^1 \sqrt{1 + x + y} dx dy \\ &= \frac{2}{3} \int_0^1 (1 + x + y)^{3/2} \Big|_{x=0}^{x=1} dy \\ &= \frac{2}{3} \int_0^1 ((2 + y)^{3/2} - (1 + y)^{3/2}) dy \\ &= \frac{2}{3} \frac{2}{5} ((2 + y)^{5/2} - (1 + y)^{5/2}) \Big|_{y=0}^{y=1} \\ &= \frac{2}{3} \frac{2}{5} (3^{5/2} - 2^{7/2} + 1). \end{aligned}$$

7 marks

9. Evaluate the integral  $\iiint_E z \, dV$ , where  $E$  is enclosed by the paraboloid  $z = x^2 + y^2$  and the plane  $z = 4$ .

Answer:  $\frac{64\pi}{3}$ 

**Solution:** Using cylindrical coordinates, we have

$$\begin{aligned}\iiint_E z \, dV &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 z \, dz \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \left( \frac{1}{2} z^2 \right)_{z=r^2}^{z=4} r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^2 (16 - r^4) r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left( 8r^2 - \frac{1}{6} r^6 \right)_{r=0}^{r=2} d\theta \\ &= \frac{1}{2} (2\pi) \left( 32 - \frac{1}{6} (64) \right) = \frac{64\pi}{3}.\end{aligned}$$

- 7 marks 10. The average value of a function  $f(x, y, z)$  on a 3-dimensional region  $E$  is given by the formula  $f_{\text{ave}} = \frac{1}{\text{Volume}(E)} \int \int \int_E f dV$ . Let  $E$  be the unit ball  $E = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ . Its volume is  $4\pi/3$ . Find the average distance from a point in  $E$  to the origin.

Answer:  $3/4$

**Solution:** Using spherical coordinates, we have

$$\begin{aligned} f_{\text{ave}} &= \frac{3}{4\pi} \int \int \int_E \sqrt{x^2 + y^2 + z^2} dV \\ &= \frac{3}{4\pi} \int_0^{2\pi} \int_0^\pi \int_0^1 \rho \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \frac{3}{4\pi} \int_0^{2\pi} d\theta \int_0^\pi \sin \phi d\phi \int_0^1 \rho^3 d\rho \\ &= \frac{3}{4\pi} (2\pi)(2)(1/4) = \frac{3}{4}. \end{aligned}$$