Final Examination — December 16, 2015 Duration: 2.5 hours

*This test has **10 questions** on **12 pages**, for a total of 80 points.*

Dr. G. Slade, Dr. C. Macdonald, Dr. B. Krause, Dr. M. Murugan

- Read all the questions carefully before starting to work. Give complete arguments and explanations for all your calculations. With the exception of #4, answers without justification will not be marked.
- Continue on the back of the *previous page* if you run out of space, *with clear indication on the original page* that your solution is continued elsewhere.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: ____________ Last Name: ____________

Student-No: ____________ Section: ____________

Signature: ____________

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	7	8	9	9	7	10	10	6	7	7	80
Score:											

1. Consider the unit sphere consisting of points (x, y, z) with $x^2 + y^2 + z^2 = 1$,

1 mark (a) Determine a normal vector to the tangent plane at a point (x, y, z) on the sphere.

Answer: $\langle 2x, 2y, 2z\rangle$

Solution: Let $F(x, y, z) = x^2 + y^2 + z^2$.
A normal vector is given by $\vec{\nabla} F = \langle 2x, 2y, 2z\rangle$.

2 marks (b) Determine the equation of the tangent plane at the point $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, and determine the equation of the tangent plane at the point $(\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$.

Answer: $x + \sqrt{3}y = 2$ and $x + \sqrt{3}z = 2$, respectively

Solution: Using the normal from part (a), the equations are

$$x + \sqrt{3}y = \frac{1}{2} + \sqrt{3}\frac{\sqrt{3}}{2} = 2$$

and

$$x + \sqrt{3}z = \frac{1}{2} + \sqrt{3}\frac{\sqrt{3}}{2} = 2.$$

2 marks (c) Determine the cosine of the acute angle θ between the two planes in part (b).

Answer: $\cos\theta = \frac{1}{4}$

Solution:

$$\cos\theta = \frac{\langle 1, \sqrt{3}, 0\rangle \cdot \langle 1, 0, \sqrt{3}\rangle}{2 \cdot 2} = \frac{1}{4}.$$

2 marks (d) Determine the equation of the line of intersection of the two planes in part (b), in symmetric form.

Answer: $\frac{x-2}{-\sqrt{3}} = y = z$

Solution: Comparing the two equations, we see that $y = z$. The first equation also gives $y = \frac{x-2}{-\sqrt{3}}$.

2. Wheat production W in a given year depends on the average temperature T and the rainfall R. It is estimated that, at current production levels, $\frac{\partial W}{\partial T} = -2$ Kt/°C (kilotonnes per Centigrade degree) and $\frac{\partial W}{\partial R} = 8$ Kt/cm (kilotonnes per cm).

4 marks (a) It is estimated that the average temperature is rising at a rate of 0.15°C/year and rainfall is decreasing at a rate of 0.1 cm/year. Using this estimated data, what is the current rate of change $\frac{\mathrm{d}W}{\mathrm{d}t}$ of wheat production (give the units too).

Answer: −1.1 Kt/year

Solution: By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t} &= \frac{\partial W}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\partial W}{\partial R}\frac{\mathrm{d}R}{\mathrm{d}t}\\ &= (-2)(0.15) + (8)(-0.1) = -0.3 - 0.8 = -1.1.\end{aligned}$$

4 marks (b) Suppose that the rainfall this year actually decreased by 0.08 cm while the average temperature increased by 0.2°C. Using differentials, estimate the actual change in production this year (give the units).

Answer: −1.04 Kt

Solution:

$$\begin{aligned}dW &= \frac{\partial W}{\partial T}dT + \frac{\partial W}{\partial R}dR\\ &= (-2)(0.2) + (8)(-0.08) = -0.4 - 0.64 = -1.04.\end{aligned}$$

3. Consider the function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ on the disk D given by $x^2+y^2 \le 4$. You may use the fact that $e \approx 2.71828$.

5 marks (a) Determine the critical points of f inside D and the values of f at those critical points.

Answer: $f(0,0) = 0$, $f(0,\pm 1) = 2e^{-1}$, $f(\pm 1, 0) = e^{-1}$.

Solution: The partial derivatives are:

$$f_x = e^{-x^2-y^2}(-2x(x^2+2y^2)+2x) = 2xe^{-x^2-y^2}(1-x^2-2y^2),$$
$$f_y = e^{-x^2-y^2}(-2y(x^2+2y^2)+4y) = 2ye^{-x^2-y^2}(2-x^2-2y^2).$$

For $f_x = 0$ we need $x = 0$ or $x^2+2y^2 = 1$, and for $f_y = 0$ we need $y = 0$ or $x^2+2y^2 = 2$. The critical points are therefore:

$$(0,0),\ (0,1),\ (0,-1),\ (1,0),\ (-1,0).$$

At these points, we have $f(0,0) = 0$, $f(0,\pm 1) = 2e^{-1}$, $f(\pm 1,0) = e^{-1}$.

3 marks (b) Determine the absolute maximum and absolute minimum values of f on the boundary of D.

Answer: maximum $f(0,\pm 2) = 8e^{-4}$, minimum $f(\pm 2, 0) = 4e^{-4}$.

Solution: On the boundary, $f(\pm\sqrt{4-y^2}, y)$ takes values $e^{-4}(4+y^2)$ with $-2 \le y \le 2$. This has maximum $f(0,\pm 2) = 8e^{-4}$ and minimum $f(\pm 2,0) = 4e^{-4}$.
Or:
Use Lagrange multipliers with $g(x) = x^2+y^2 = 4$ and solve:

$$\lambda 2x = 2xe^{-x^2-y^2}(1-x^2-2y^2),$$
$$\lambda 2y = 2ye^{-x^2-y^2}(2-x^2-2y^2).$$

Cancelling $2x$ in first equation and $2y$ in second leads to inconsistent equations. We get a solution from $x = 0$ and $0^2+y^2 = 4$, namely $(0,\pm 2)$. We get a solution also from $y = 0$ and $x^2+0^2 = 4$, namely $(\pm 2, 0)$.
So again we get maximum $f(0,\pm 2) = 8e^{-4}$ and minimum $f(\pm 2,0) = 4e^{-4}$.

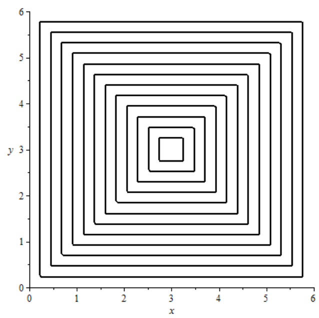
1 mark (c) What are the locations and values of the absolute maximum and absolute minimum of f on D?

Answer: minimum $f(0,0) = 0$, maximum $f(0,\pm 1) = 2e^{-1}$.

Solution: We seek the smallest and largest values among: $0, 4e^{-4}, 8e^{-4}, e^{-1}, 2e^{-1}$. They are listed here in order.

9 marks

4. Consider the following 9 contour plots and 9 graphs (next page). Each contour plot is the contour plot of one of the 9 graphs. Match each contour plot with the corresponding graph. In the 9 contour plots, the x axis is horizontal, the y axis is vertical and the *values* of the contours are evenly spaced. Write the number corresponding to the matching graph next to the letter labelling each contour plot.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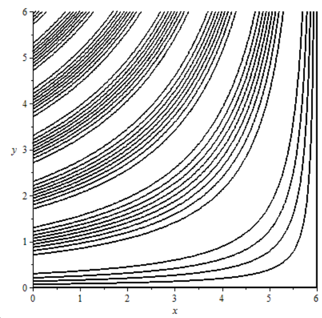
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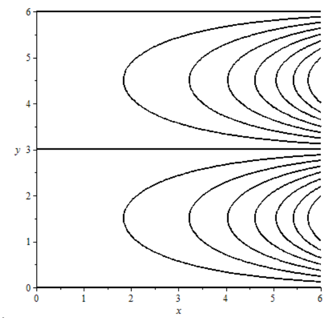
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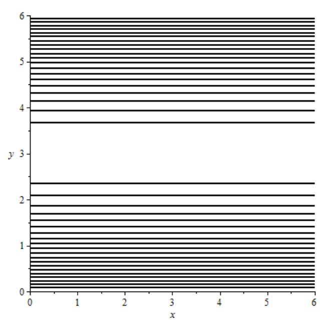
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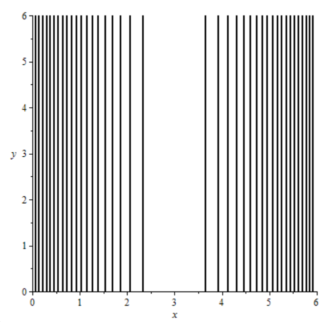
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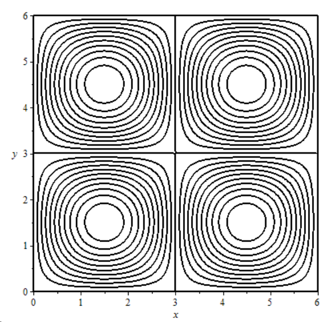
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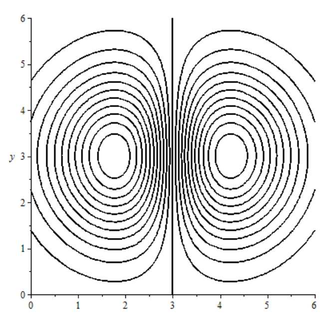
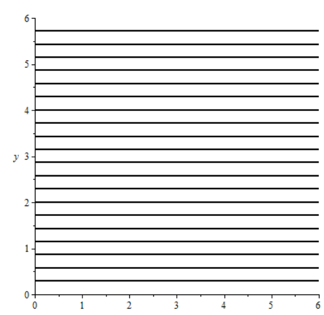
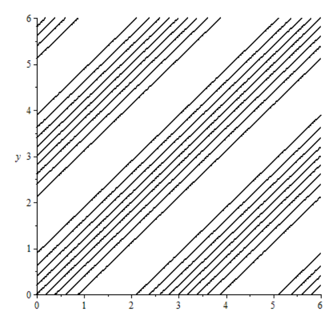

G.

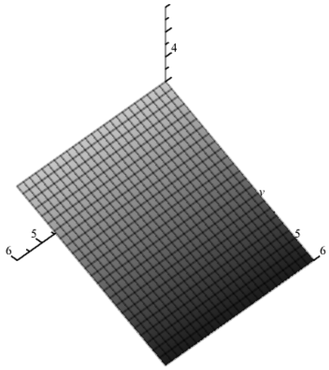

H.


I.

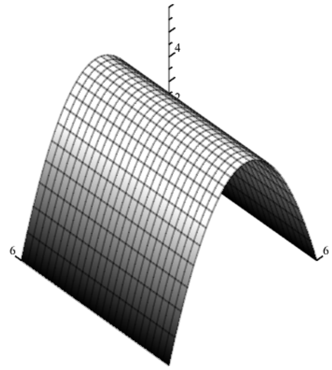
Solution: A4, B9, C7, D3, E2, F5, G8, H1, I6

In the 9 graphs below, the positive x axis is on the left, the positive y axis is on the right, and the positive z axis is upward.

Nothing written on this page will be marked.



1.



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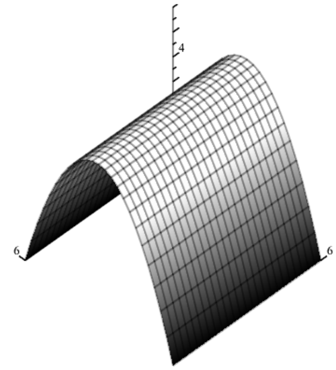

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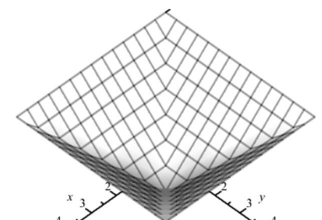

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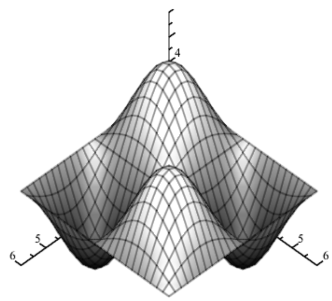

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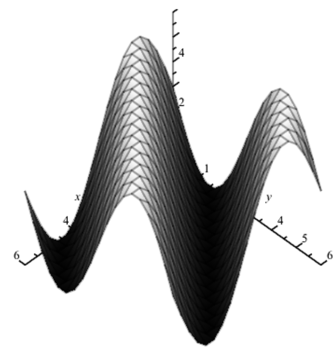

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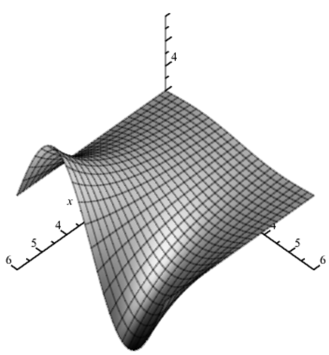

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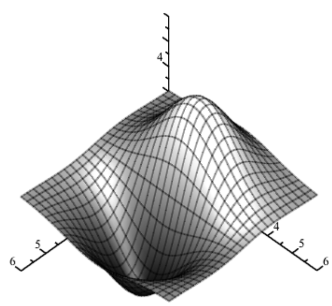
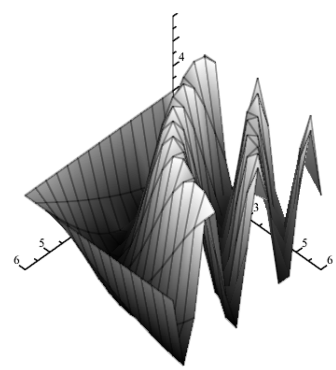


8.



9.

5. Consider a hill whose height is described by $f(x,y) = 100 - \frac{1}{2}x^2 - \frac{1}{2}y^2$, measured in metres.

3 marks

(a) At time $t = 0$, I start walking on the hill at position $(5, 5, 75)$. I am walking at 1 metre/sec, and I set out in the direction $\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$. At what rate is my altitude changing at time 0?

Answer: $-\frac{5}{2}(1 + \sqrt{3})$ m/s

Solution: First, $\vec{\nabla} f = \langle -x, -y \rangle$, so $\vec{\nabla} f(5,5) = \langle -5, -5 \rangle$. Let $\vec{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$. My initial rate of change is the directional derivative

$$D_{\vec{u}} f(5,5) = \vec{\nabla} f(5,5) \cdot \vec{u} = \langle -5, -5 \rangle \cdot \langle 1, \sqrt{3} \rangle \frac{1}{2} = -\frac{5}{2}\left(1 + \sqrt{3}\right).$$

1 mark

(b) Find an upward-pointing normal vector $\vec{n}$ to the surface of the hill.

Answer: $\vec{n} = \langle x, y, 1 \rangle$

Solution: Let $F(x,y,z) = z - f(x,y) = z + \frac{1}{2}x^2 + \frac{1}{2}y^2 - 100$.
An upward-pointing normal is $\vec{\nabla} F = \langle x, y, 1 \rangle$.

3 marks

(c) Douglas fir trees grow vertically (in the z-direction) on the surface of the hill. Where on the hill is the angle α between the tree trunks and the normal vector given by $\alpha = \frac{\pi}{3}$?

Answer: On the hill directly above the circle of radius $\sqrt{3}$ centred at the origin of the x-y plane.

Solution: The trees point upwards in the $\hat{k}$ direction. We want

$$\cos\alpha = \frac{1}{2} = \frac{\vec{\nabla} F \cdot \hat{k}}{|\vec{\nabla} F|\, |\hat{k}|} = \frac{\langle x, y, 1 \rangle \cdot \langle 0, 0, 1 \rangle}{|\vec{\nabla} F|} = \frac{1}{|\vec{\nabla} F|} = \frac{1}{\sqrt{x^2 + y^2 + 1}}.$$

That is

$$\begin{aligned} \sqrt{x^2 + y^2 + 1} &= 2 \\ x^2 + y^2 + 1 &= 4 \\ x^2 + y^2 &= 3 \end{aligned}$$

6. Consider the integral

$$\int_0^2 \int_{y^2}^4 y^3 e^{x^3}\, dx\, dy.$$

5 marks

(a) Write the integral in reversed order.

Answer: $\int_0^4 \int_0^{\sqrt{x}} y^3 e^{x^3}\, dy\, dx$

Solution:

$$\int_0^2 \int_{y^2}^4 y^3 e^{x^3}\, dx\, dy = \int_0^4 \int_0^{\sqrt{x}} y^3 e^{x^3}\, dy\, dx.$$

5 marks

(b) Using the result of part (a), evaluate the integral.

Answer: $\frac{1}{12}(e^{64} - 1)$

Solution:

$$\begin{aligned}\int_0^4 \int_0^{\sqrt{x}} y^3 e^{x^3}\, dy\, dx &= \int_0^4 \left[\frac{y^4}{4} e^{x^3}\right]_{y=0}^{y=\sqrt{x}} dx \\ &= \int_0^4 \frac{x^2}{4} e^{x^3}\, dx.\end{aligned}$$

Then we make the substitution $u = x^3$ to evaluate

$$\int_0^4 \frac{x^2}{4} e^{x^3}\, dx = \int_0^{64} \frac{1}{12} e^u\, du = \frac{1}{12} \left[e^u\right]_{u=0}^{u=64} = \frac{e^{64} - 1}{12}.$$

7. Consider the wedge-shaped region contained inside the cylinder $x^2 + y^2 = 9$, bounded above by the plane $z = x$, and bounded below by the xy plane.

5 marks (a) Write a double integral (including limits of integration) whose value is the volume of the wedge-shaped region.

Answer:
$V = \int_{-\pi/2}^{\pi/2} \int_0^3 r\cos\theta\, r\, dr\, d\theta$

Solution:

$$V = \int_{-\pi/2}^{\pi/2} \int_0^3 r\cos\theta\, r\, dr\, d\theta.$$

Or:

$$V = \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x\, dy\, dx.$$

5 marks (b) Evaluate the integral in part (a) to determine the volume of the wedge-shaped region.

Answer: 18

Solution:

$$\begin{aligned} V &= \int_{-\pi/2}^{\pi/2} \int_0^3 r\cos\theta\, r\, dr\, d\theta \\ &= \int_{-\pi/2}^{\pi/2} \cos\theta \frac{1}{3} r^3 \Big|_0^3 d\theta \\ &= 9 \int_{-\pi/2}^{\pi/2} \cos\theta d\theta = 9\sin\theta \Big|_{-\pi/2}^{\pi/2} = 18. \end{aligned}$$

Or:

$$\begin{aligned} V &= \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x\, dy\, dx \\ &= \int_0^3 x 2\sqrt{9-x^2}\, dx \\ &= -\int_9^0 u^{1/2}\, du \quad (u = 9 - x^2) \\ &= \frac{2}{3} u^{3/2} \Big|_0^9 = 18. \end{aligned}$$

6 marks

8. Determine the surface area of the surface given by $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, over the square $0 \le x \le 1$, $0 \le y \le 1$.

Answer: $\frac{4}{15}(3^{5/2} - 1)$

Solution: Since $\frac{\partial z}{\partial x} = x^{1/2}$ and $\frac{\partial z}{\partial y} = y^{1/2}$,

$$\begin{aligned}
A &= \int_0^1 \int_0^1 \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy \\
&= \int_0^1 \int_0^1 \sqrt{1+x+y}\, dxdy \\
&= \frac{2}{3}\int_0^1 (1+x+y)^{3/2}\Big|_{x=0}^{x=1}\, dy \\
&= \frac{2}{3}\int_0^1 \left((2+y)^{3/2} - (1+y)^{3/2}\right)\, dy \\
&= \frac{2}{3}\frac{2}{5}\left((2+y)^{5/2} - (1+y)^{5/2}\right)_{y=0}^{y=1} \\
&= \frac{2}{3}\frac{2}{5}\left(3^{5/2} - 2^{7/2} + 1\right).
\end{aligned}$$

7 marks

9. Evaluate the integral $\int\int\int_E z\, dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Answer: $\frac{64\pi}{3}$

Solution: Using cylindrical coordinates, we have

$$\begin{aligned}
\int\int\int_E z\, dV &= \int_0^{2\pi}\int_0^2\int_{r^2}^4 z\, dz\, r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^2 \left(\frac{1}{2}z^2\right)\Big|_{z=r^2}^{z=4} r\, dr\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^2 \left(16 - r^4\right) r\, dr\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(8r^2 - \frac{1}{6}r^6\right)\Big|_{r=0}^{r=2} d\theta \\
&= \frac{1}{2}(2\pi)\left(32 - \frac{1}{6}(64)\right) = \frac{64\pi}{3}.
\end{aligned}$$

7 marks 10. The average value of a function $f(x, y, z)$ on a 3-dimensional region E is given by the formula $f_{\text{ave}} = \frac{1}{\text{Volume}(E)} \int\int\int_E f dV$. Let E be the unit ball $E = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$. Its volume is $4\pi/3$. Find the average distance from a point in E to the origin.

Answer: 3/4

Solution: Using spherical coordinates, we have

$$\begin{aligned} f_{\text{ave}} &= \frac{3}{4\pi} \int\int\int_E \sqrt{x^2 + y^2 + z^2}\, dV \\ &= \frac{3}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho\rho^2 \sin\phi\ d\rho d\phi d\theta \\ &= \frac{3}{4\pi} \int_0^{2\pi} d\theta \quad \int_0^{\pi} \sin\phi d\phi \quad \int_0^1 \rho^3 d\rho \\ &= \frac{3}{4\pi}(2\pi)(2)(1/4) = \frac{3}{4}. \end{aligned}$$