1. Consider the surface described by the equation $e^{xyz-1} + 3 = x^2 + y^2 + z^2$.

4 marks

(a) Find a function $F(x, y, z)$ such that (a, b, c) is on the surface if and only if we have $F(a, b, c) = 0$. Compute the gradient of F.

Solution:
It suffices to take $F(x, y, z) = e^{xyz-1} + 3 - x^2 - y^2 - z^2$ or $F(x, y, z) = x^2 + y^2 + z^2 - e^{xyz-1} - 3$.

$$\frac{\partial F}{\partial x}(x, y, z) = yze^{xyz-1} - 2x$$
$$\frac{\partial F}{\partial y}(x, y, z) = xze^{xyz-1} - 2y$$
$$\frac{\partial F}{\partial z}(x, y, z) = xye^{xyz-1} - 2z$$

Thus $\nabla F(x, y, z) = \langle yze^{xyz-1} - 2x, xze^{xyz-1} - 2y, xye^{xyz-1} - 2z\rangle$ (or - that value).

4 marks

(b) Give the equation of the tangent plane to the surface at $(x_0, y_0, z_0) = (1, 1, 1)$ in the form $x + by + cz + d = 0$. Note that we require the coefficient next to x to be 1.

Solution: The tangent plane is given by $\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$. We have $\nabla F(1, 1, 1) = (-1, -1, -1)$. Therefore we obtain

$$-(x - 1) - (y - 1) - (z - 1) = 0$$

And hence the equation is $x + y + z = 3$.

2. The sugar concentration in an infinite 3D compost bin is given by the equation $S(x, y, z) = (x + y + 2z)^2$. A fruit fly is at position $(1, 1, 1)$.

2 marks

(a) Compute the directional derivative of S at $(1, 1, 1)$ in the direction of $\vec{u} = (-1, 0, 1)$.

Solution: Note that $\nabla S(x, y, z) = 2(x+y+2z)\langle 1, 1, 2\rangle$. In particular $\nabla S(1, 1, 1) = \langle 8, 8, 16\rangle$. Therefore the directional derivative is

$$D_{\vec{u}}S(1, 1, 1) = \langle 8, 8, 16\rangle \cdot \langle -1, 0, 1\rangle/\sqrt{2}$$

which gives $D_{\vec{u}}S(1, 1, 1) = \frac{8}{\sqrt{2}}$.
Note: If you forgot to divide by the norm of $\vec{u}$ you get no marks.

2 marks

(b) The fruit fly is feeling a little hungry. In what (unit) direction should the fly move if it wishes to increase the concentration of sugar in the fastest possible way?

Solution: In the direction of the gradient $\nabla S(1, 1, 1) = \langle 8, 8, 16\rangle$. Therefore the unit direction is $\langle 1, 1, 2\rangle/\sqrt{6}$.

2 marks

(c) The fly is now happy with the amount of sugar in its position. Give a (unit) direction in which the fly could move if it wishes to keep the concentration of sugar constant.

Solution: Any direction orthogonal to the gradient works. For instance

$$\vec{u} = \langle 1, 1, -1 \rangle / \sqrt{3}.$$

6 marks

3. (a) A differentiable function $z = f(x, y)$ is unknown, but an alien supercomputer gave us precise values of $f(x, y)$ and its derivatives on points A, B, C and D.

point	f	f_x	f_y	f_{xx}	f_{yy}	f_{xy}
A	1	0	0	1	0	-5
B	1	0	-2	3	8	4
C	2	0	0	3	3	-2
D	2	0	0	3	3	6

For points A, B, C and D determine whether they are a local minimum, local maximum, a saddle point, or none of the above.

Solution: A, C and D are critical points because $\nabla f = 0$, whereas B is not a critical point. In each case we compute $f_{xx} f_{yy} - f_{xy}^2$. In A the value is negative, thus A is a saddle point. In C it is positive and f_{xx} is positive, thus it is a local min. in D the value is negative, thus it is a saddle point.

- A is: a saddle point.
- B is: none of the above.
- C is: a local minimum.
- D is: a saddle point.

2 marks

(b) **(Bonus marks)** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that $f(1, 0) = f(0, 0) = 0$. Show that there exists $\langle a, b \rangle$ such that $\nabla f(a, b)$ is orthogonal to $\langle 1, 0 \rangle$. *Hint:* Define $g(t) = f(t, 0)$. Combine the 1D mean value theorem and the chain rule to conclude.

Solution: Note that $g(0) = f(0, 0) = 0$ and $g(1) = f(1, 0) = 0$. By the mean value theorem, there is $c \in (0, 1)$ such that $g'(c) = 0$. On the other hand, $g'(c) = f_x(c, 0) \cdot 1 + f_y(c, 0) \cdot 0 = \nabla f(c, 0) \cdot \langle 1, 0 \rangle$. Putting this together we obtain that $\nabla f(c, 0) \cdot \langle 1, 0 \rangle = 0$ and thus $\nabla f(c, 0)$ is orthogonal to $\langle 1, 0 \rangle$. Therefore setting $\langle a, b \rangle = \langle c, 0 \rangle$ works.