1. Consider the function $f(x,y) = x^2y + e^{x-y}$ for parts (a),(b),(c) and (d).

4 marks (a) Compute f_x, f_y, f_{xy} and f_{yx}.

Solution:

$$f_x = 2xy + e^{x-y}$$
$$f_y = x^2 - e^{x-y}$$
$$f_{xy} = f_{yx} = 2x - e^{x-y}$$

4 marks (b) Compute the equation of the tangent plane to the graph of $z = f(x,y)$ at $(2,2,9)$.

Solution: The equation is given by

$$z = f(2,2) + f_x(2,2)(x-2) + f_y(2,2)(y-2)$$

Thus,

$$z = 9 + 9(x-2) + 3(y-2)$$

Simplifying, we get

$$z = 9x + 3y - 15$$

2 marks (c) **Use the previous part** to approximate $f(2.1, 1.9)$.

Solution:

$$\begin{aligned} f(2.1,1.9) &\approx 9 + 9(2.1-2) + 3(1.9-2) \\ &= 9 + 0.9 - 0.3 \\ &= 9.6 \end{aligned}$$

For reference, the actual value is approx 9.60040275816017

4 marks (d) Find a point (a, b, c) in the graph of $z = f(x, y)$ such that its tangent plane has the equation $3x - z = 1$. *Hint: there is a solution such that $a = 1$.*

Solution: The general equation of the tangent plane at $(a, b, f(a, b))$ is

$$z = f(a, b) + (2ab + e^{a-b})(x - a) + (a^2 - e^{a-b})(y - b)$$

Rearranging we get:

$$(2ab + e^{a-b})x + (a^2 - e^{a-b})y - z = -f(a, b) + b(a^2 - e^{a-b}) + a(2ab + e^{a-b})$$

We must have $a^2 - e^{a-b} = 0$ and $2ab + e^{a-b} = 3$. Following the hint we impose $a = 1$. We must now have $1 - e^{1-b} = 0$ and $2 + e^{1-b} = 3$. The only possibility is $b = 1$ which works. Replacing $(a, b) = (1, 1)$ we get:

$$3x - z = -f(1, 1) + 3 = 1$$

Which satisfies the requirements.

6 marks 2. (a) You maneuver a spaceship in a three dimensional space. At time t the position of the spaceship is given by the vector $\langle x(t), y(t), z(t)\rangle$. A proton star at the origin emits radiation in such a way that the perceived radiation at a point in space is given by the equation $R(x, y, z) = e^{-(x^2+y^2+z^2)}$.

Assume that at time $t = 0$ the position of the spaceship is $\langle x(0), y(0), z(0)\rangle = \langle 1, 1, 1\rangle$ and that its velocity is $\langle x'(0), y'(0), z'(0)\rangle = \langle 1, 2, -4\rangle$. Determine the rate of change of the perceived radiation by the spaceship at time $t = 0$.

Solution: Note that $R_x = -2xe^{-(x^2+y^2+z^2)}$, $R_y = -2ye^{-(x^2+y^2+z^2)}$ and $R_z = -2ze^{-(x^2+y^2+z^2)}$. In particular $R_x(1, 1, 1) = R_y(1, 1, 1) = R_z(1, 1, 1) = -2e^{-3}$. By the chain rule,

$$\begin{aligned} R'(x(0), y(0), z(0)) &= R_x(1, 1, 1)x'(0) + R_y(1, 1, 1)y'(0) + R_z(1, 1, 1)z'(0) \\ &= -2e^{-3}(1 + 2 - 4) = 2e^{-3}. \end{aligned}$$

2 marks (b) **(Bonus marks)** Construct an example of a function $f(x, y)$ such that every level curve is a single line of the form $c = 2x + y$ for some $c \in \mathbb{R}$ but whose graph is NOT a plane.

Solution: $f(x, y) = (2x + y)^3$

Note that $(2x + y)^2$ does not work, as the contour curves have two branches. Any odd power different than 1 also works.