Sturmian configurations through asymptotic pairs

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Low Complexity Dynamical Systems - BRINMRC

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- Let A be a finite set and $d \geq 1$ be an integer.
- A **configuration** is a map $x\colon \mathbb{Z}^d \to A$.

Let σ denote the $\mathbb{Z}^d$ **shift action** on $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

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We say two configurations $x, y \in A^{\mathbb{Z}^d}$ are **asymptotic** if there exists a finite $F \subset \mathbb{Z}^d$ such that $x|_{\mathbb{Z}^d \setminus F} = y|_{\mathbb{Z}^d \setminus F}$.

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- The smallest such $F = \{n \in \mathbb{Z}^d : x(n) \neq y(n)\}$ is their **difference set**.

x, y are asymptotic if and only if for any sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	0	2	2	1	0	2	1	0	2
2	1	0	2	1	0	2	1	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

$$F = \{(0,0), (-1,0), (0,-1)\}.$$

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- Given $S \Subset \mathbb{Z}^d$ and a pattern $p \in A^S$ let

$$[p] = \{z \in A^{\mathbb{Z}^d} : z|_S = p\}.$$

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$$\Delta_p(x, y) = \sum_{u \in \mathbb{Z}^d} 1_{[p]}(\sigma^u(y)) - 1_{[p]}(\sigma^u(x)).$$

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$$\Delta_p(x, y) = \sum_{u \in \boxed{F - S}} 1_{[p]}(\sigma^u(y)) - 1_{[p]}(\sigma^u(x)).$$

We say an asymptotic pair x, y is indistinguishable if $\Delta_p(x, y) = 0$ for every pattern p.

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
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So for every pattern p with support S, we have $\Delta_p(x, y) = 0$.

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- (x, x) for any $x \in A^{\mathbb{Z}^d}$ is an indistinguishable asymptotic pair. We call it **trivial**.

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- (x, x) for any $x \in A^{\mathbb{Z}^d}$ is an indistinguishable asymptotic pair. We call it **trivial**.
- If $x, y \in A^{\mathbb{Z}^d}$ are asymptotic and on the same orbit ($\sigma^n(y) = x$ for some $n \in \mathbb{Z}^d$) then they are indistinguishable.

x

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	1	0	0
0	0	0	0	1	1	1	1	1	0
0	0	0	0	1	1	1	1	1	0
0	0	0	0	0	1	1	1	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

y

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	0	0	0	0
0	0	1	1	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Do there exist indistinguishable asymptotic pairs which are not on the same orbit?

Origin of the question

Consider n balls with real weights given by a map f.

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots \quad n$$

$$f(1) \quad f(2) \quad f(3) \quad f(4) \quad f(5) \quad f(6) \quad f(7) \qquad f(n)$$

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$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$f(7)$		$f(n)$

¿What is the probability distribution $\mu = (\mu_1, \ldots, \mu_n)$ on $\{1, \ldots, n\}$ that maximizes entropy plus average weight?

$$\max_{\mu} \left(H(\mu) + \int f \, \mathrm{d}\mu \right) = \max_{\mu} \sum_{i=1}^{n} (-\mu_i \log(\mu_i) + f(i)\mu_i) \, .$$

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Answer: Boltzmann's distribution.

$$\mu_k = \frac{\exp(f(k))}{\sum_{i=1}^{n} \exp(f(i))}.$$

Gibbs Measures

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Denote the set of all asymptotic pairs (x, y) by $\mathcal{A}$. The Boltzmann distribution of a Gibbs measure is determined by a **cocycle** $\Psi\colon \mathcal{A} \to \mathbb{R}$, that is, a map which satisfies:

$$\Psi(x, y) = \Psi(x, z) + \Psi(z, y) \text{ for all } (x, y), (y, z) \in \mathcal{A}.$$

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The space of continuous, shift-invariant cocycles $\mathcal{B}$ is a Banach space with an appropriate norm.

1. There is a natural evaluation map on $\mathcal{B}^*$. For $(x, y) \in \mathcal{A}$ we have $\mathrm{ev}_{x,y} \in \mathcal{B}^*$ given by

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2. It can be shown that the strong norm on $\mathcal{B}^*$ for these evaluation maps is given by

$$\|\mathrm{ev}_{x,y}\| = \sup_{S \in \mathbb{Z}^d} \frac{1}{|S|} \sum_{p \in A^S} |\Delta_p(x, y)|.$$

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3. **An asymptotic pair gives the trivial linear functional if and only if it is indistinguishable.**

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Theorem (SB + SL + ŠS, 2021)

Yes!

We completely characterize them on $\mathbb{Z}$. They are closely connected to Sturmian codings of irrational rotations.

Basic properties of indistinguishable pairs:

Recall: (x, y) is indistinguishable if and only if $\Delta_p(x, y) = 0$ for every $S \Subset \mathbb{Z}^d$ and pattern $p \in A^S$.

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In particular, it suffices to check the property on rectangular patterns (or words in the case of $\mathbb{Z}$).

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Indistinguishable asymptotic pairs are invariant under actions of the affine group of $\mathbb{Z}^d$.

In particular, they are invariant under the shift map.

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$(x_n, y_n)_{n\in\mathbb{N}}$ **converges in the asymptotic relation** to (x, y) if

- $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ converge to x, y respectively.
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- $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ converge to x, y respectively.
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If $(x_n, y_n)_{n\in\mathbb{N}}$ converges in the asymptotic relation to (x, y) and every pair (x_n, y_n) is indistinguishable, then (x, y) is indistinguishable.

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Let (x, y) be a non-trivial indistinguishable asymptotic pair. If x is not recurrent, then x and y lie on the same orbit.

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4. As $\bigcap_{n\in\mathbb{N}}[p_n] = \sigma^k(x)$, we conclude that $\sigma^k(x) = \sigma^m(y)$.

The case of $\mathbb{Z}$

On $\mathbb{Z}$ life is easier (as opposed to $\mathbb{Z}^d$ with $d \geq 2$):

Let (x, y) be a non-trivial indistinguishable asymptotic pair. If a pattern p occurs in x, then it occurs intersecting their difference set.

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Corollary: If x, y are indistinguishable with difference set $F = [\![0, k-1]\!]$ then their word complexity satisfies

$$|\mathcal{L}_n(x)| \leq k + n - 1.$$

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Thus x, y must be Sturmian configurations!

Formally, given $\alpha \in [0,1] \setminus \mathbb{Q}$ let $c_\alpha, c'_\alpha \in \{0,1\}^{\mathbb{Z}}$ be given by

$$c_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor.$$
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That is, the codings of the orbit of 0 under rotation by α in the circle $\mathbb{R}/\mathbb{Z}$ with partitions $\mathcal{P} = \{[0, 1-\alpha), [1-\alpha, 1)\}$ and $\mathcal{P}' = \{(0, 1-\alpha], (1-\alpha, 1]\}$ respectively.
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The pair (c_α, c'_α) is indistinguishable. In fact, every pattern in their language occurs exactly once intersecting each of their difference sets.

Theorem: B, Labbé and Starosta

Let $x, y \in \{0,1\}^{\mathbb{Z}}$ and assume that x is recurrent. The following are equivalent:

- (x, y) is an indistinguishable asymptotic pair with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$
- There exists $\alpha \in [0,1] \setminus \mathbb{Q}$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the lower and upper characteristic Sturmian sequences of slope α.

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The non-recurrent case is an asymptotic limit of Sturmians.

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But there is more...

The general case can be obtained from Sturmians using shifts and substitutions.

Theorem: B, Labbé and Starosta

Let A be a finite alphabet and $x, y \in A^{\mathbb{Z}}$ a non-trivial asymptotic pair. Then x, y is indistinguishable if and only if either

- x is recurrent and there exists $\alpha \in [0,1] \setminus \mathbb{Q}$, a substitution $\varphi\colon \{0,1\} \to A^+$ and an integer $m \in \mathbb{Z}$ such that

$$\{x, y\} = \{\sigma^m \varphi(\sigma(c_\alpha)), \sigma^m \varphi(\sigma(c'_\alpha))\},$$

- x is not recurrent and there exists a substitution $\varphi\colon \{0,1\} \to A^+$ and an integer $m \in \mathbb{Z}$ such that

$$\{x, y\} = \{\sigma^m \varphi({}^\infty 0.10^\infty), \sigma^m \varphi({}^\infty 0.010^\infty)\}.$$

What about $d \geq 2$?

Things are much harder:

- Patterns may occur without intersecting the difference set.
- recurrent indistinguishable pairs may not be uniformly recurrent.
- Substitutions do not help reduce the problem to a small size difference set (no good notion of derived sequences).
- In general, there is no complexity bound.

Example:

x

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	0	1	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

y

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	1	0	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

The horizontal configuration is a 1-dimensional indistinguishable pair, everything else is the symbol 2.

Example:

x

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	0	1	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

y

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	1	0	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

The horizontal configuration is a 1-dimensional indistinguishable pair, everything else is the symbol 2.

- Recurrent but not uniformly recurrent.
- Some patterns do not occur in the difference set.

Theorem: B and Labbé.

Let $d \geq 1$ and $x, y \in \{0, \ldots, \mathtt{d}\}^{\mathbb{Z}^d}$ be an asymptotic pair with difference set $F = \{0, -e_1, \ldots, -e_d\}$. TFAE:

1. The asymptotic pair (x, y) is indistinguishable, satisfies the **flip condition** and x is uniformly recurrent.
2. There exists a totally irrational vector $\alpha \in [0, 1)^d$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the **characteristic multidimensional Sturmian configurations** of slope α.

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1. The asymptotic pair (x, y) is indistinguishable.
2. For every nonempty finite connected subset $S \subset \mathbb{Z}^d$ and $p \in \mathcal{L}_S(x) \cup \mathcal{L}_S(y)$, p intersects the difference set F exactly once in both x and y.
3. For every nonempty finite connected subset $S \subset \mathbb{Z}^d$, we have
$$|\mathcal{L}_S(x)| = |\mathcal{L}_S(y)| = |F - S|.$$
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Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d$, let $\tau \in S_d$ such that

$$1 \geq \alpha_{\tau(1)} \geq \alpha_{\tau(2)} \geq \cdots \geq \alpha_{\tau(d)} \geq 0.$$

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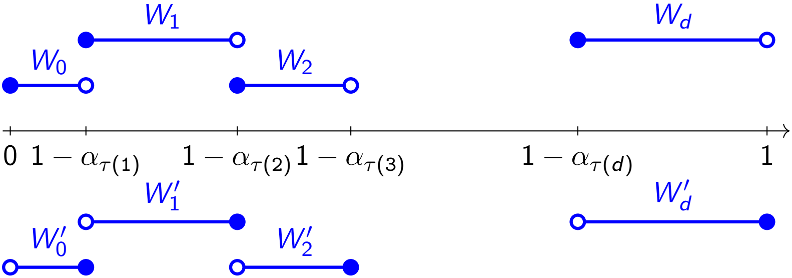
Then the partitions $\mathcal{W}$ and $\mathcal{W}'$ are given by:



Explicitly, given $\alpha = (\alpha_1, \dots, \alpha_d)$ we have

$$\begin{aligned} c_\alpha : \quad \mathbb{Z}^d &\to \{\texttt{0}, \dots, \texttt{d}\} \\ n &\mapsto \sum_{i=1}^{d} \left(\lfloor \alpha_i + n \cdot \alpha \rfloor - \lfloor n \cdot \alpha \rfloor \right), \end{aligned}$$

and

$$\begin{aligned} c'_\alpha : \quad \mathbb{Z}^d &\to \{\texttt{0}, \dots, \texttt{d}\} \\ n &\mapsto \sum_{i=1}^{d} \left(\lceil \alpha_i + n \cdot \alpha \rceil - \lceil n \cdot \alpha \rceil \right). \end{aligned}$$

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The configurations c_α, c'_α are asymptotic with difference set $F = \{0, -e_1, \ldots, -e_d\}$.

Recall the picture from the beginning:

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	0	2	2	1	0	2	1	0	2
2	1	0	2	1	0	2	1	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

We have $x = c_\alpha$ and $y = c'_\alpha$ respectively for

$$\alpha = \left(\frac{\sqrt{2}}{2}, \sqrt{19} - 4\right).$$

Flip Condition

Let $x, y \in \{0, \ldots, \mathtt{d}\}^{\mathbb{Z}^d}$ be an asymptotic pair. We say it satisfies the **flip condition** if:

1. the difference set of x and y is $F = \{0, -e_1, \ldots, -e_d\}$,
2. the restriction $x|_F$ is a bijection $F \to \{0, \ldots, \mathtt{d}\}$ such that $x_0 = 0$,
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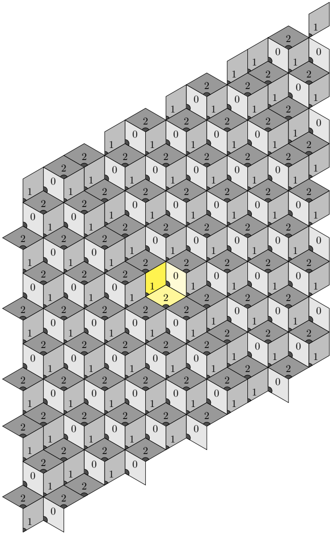
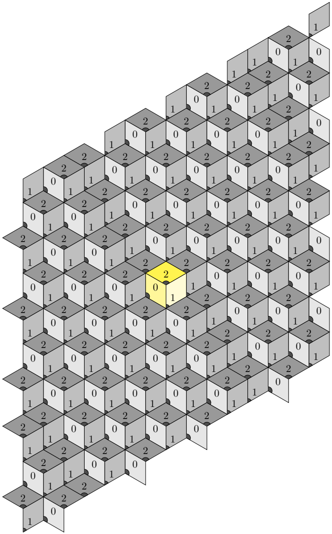
The conditions above induce a permutation on $\{0, \ldots, d\}$ defined by $y_n \mapsto x_n$ for every $n \in F$, which is the cyclic permutation $(0, 1, \ldots, d)$ of the alphabet.

The flip condition can be interpreted as flipping the unit hypercube on a co-dimension 1 discrete subspace.

x



y



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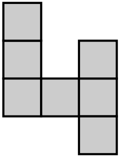
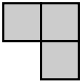
Say $c_\alpha \in \{0, 1, 2\}^{\mathbb{Z}^d}$ and you need to know how many patterns with support $S \Subset \mathbb{Z}^2$ there are.

$S =$ 

$F =$ 

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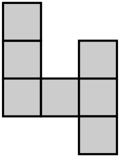
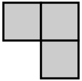
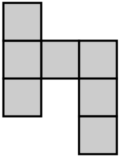
$S =$ 

$F =$ 

$F - S =$ 

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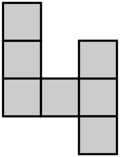
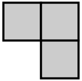
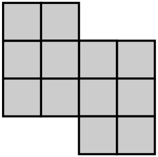
$S =$  $F =$  $F - S =$ 

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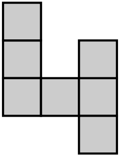

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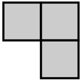

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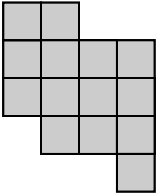

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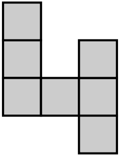
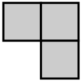
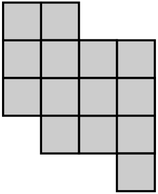
$S =$  $F =$  $F - S =$ 

There are exactly 14 patterns with support S on a 2-dimensional Sturmian configuration.

Let $(m_1, \ldots, m_d) \in \mathbb{N}^d$ and consider the box

$$B = \prod_{i=1}^{d} [\![0, m_i - 1]\!].$$

In this case we get a beautiful formula for the complexity of a multidimensional Sturmian configuration x:

$$|\mathcal{L}_B(x)| = |\mathcal{L}_{(m_1,\ldots,m_d)}(x)| = m_1 \cdots m_d \left(1 + \frac{1}{m_1} + \cdots + \frac{1}{m_d}\right).$$

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$\triangleright$ For $d = 1$ we recover $\mathcal{L}_n(x) = n + 1$.

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1. Maybe the $\mathbb{Z}^d$ setting is completely wrong. Redefine everything for Delone sets in $\mathbb{R}^d$ and use ad-hoc tools from that setting (in progress in joint work with Sébastien Labbé).
2. Most of the basic properties hold for arbitrary countable groups. Are there natural properties that would generate interesting "Sturmian-like" configurations on groups?

Thanks!

Indistinguishable asymptotic pairs and multidimensional Sturmian configurations.
S. Barbieri, S. Labbé
https://arxiv.org/abs/2204.06413

A characterization of Sturmian sequences by indistinguishable asymptotic pairs
S. Barbieri, S. Labbé, Š. Starosta
https://doi.org/10.1016/j.ejc.2021.103318