Effective Dynamics

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UT Groups & Dynamics seminar

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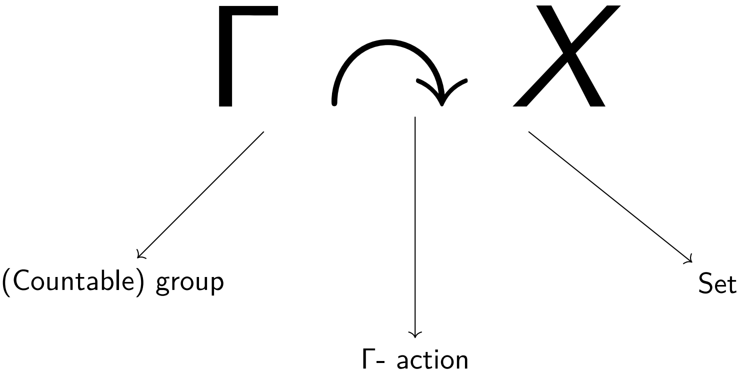
What's a dynamical system?

What's a dynamical system?

Answer: https://arxiv.org/list/math.DS/recent

Dynamical systems

Dynamical system



Dynamical systems

Topological Dynamical Systems

$$\Gamma \curvearrowright X$$

(Countable) group

Γ- action by homeomorphisms

Topological space
(compact metrizable)

Dynamical systems

Algebraic Dynamical Systems

$$\Gamma \curvearrowright X$$

(Countable) group

Compact (abelian) group

Γ- action by automorphisms

Measurable Dynamical Systems
(ergodic theory)

$$\Gamma \curvearrowright (X, \mu)$$

(Countable) group

Probability measure preserving action

Standard Borel space

Borel probability measure

Effective dynamical systems

Effective Dynamical Systems

$$\Gamma \curvearrowright X$$

(Finitely generated) group

Effective Γ- action

Effective space

Effective dynamical systems

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Effective space

Effective ↔ "Can be described through an algorithm"

Algorithms

▷ **Informal:** An algorithm is a list of instructions that are applied sequentially.

- Computer program.
- Cooking recipe.

GCD

On input $a, b \in \mathbb{N}$:
  if $b = 0$:
    return a;
  else:
    return GCD$(b, a \bmod b)$;

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GCD

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▷ **Formal:** Turing machine.

Turing machines

A Turing machine T is given (essentially) by:

- A finite set Σ (alphabet).
- A finite set Q (states).
- A map $\delta_T \colon \Sigma \times Q \to \Sigma \times Q \times \{-1, 0, 1\}$ (transition function).

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Additionally it has:

- A special blank symbol $\sqcup$.
- Some extra “auxiliary” symbols $\Sigma' \ni \sqcup$.

$$\delta_T \colon (\Sigma \cup \Sigma') \times Q \to (\Sigma \cup \Sigma') \times Q \times \{-1, 0, 1\}.$$

- An initial state $q_0 \in Q$.
- A halting state $q_H \in Q$.

Turing machines

Turing machines induce a map on the space

$$\Sigma^{\mathbb{Z}} \times Q \times \mathbb{Z}$$

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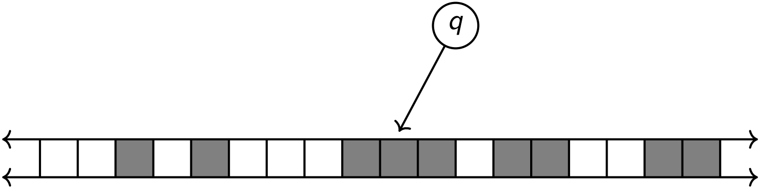
Example:

- $\Sigma = \{\square, \blacksquare\}$.



$$\delta_T(\blacksquare, q) = (\square, r, -1)$$

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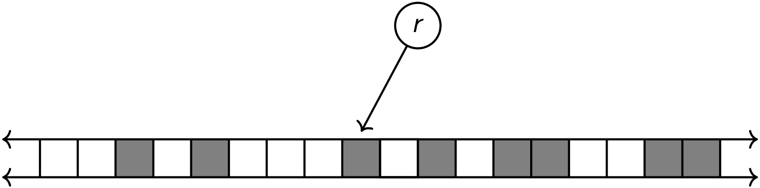

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Turing machines

Let $w_0 \dots w_{n-1} \in \Sigma^n$ and consider $\widetilde{w} \in (\Sigma \cup \Sigma')^{\mathbb{Z}}$ given by

$$\widetilde{w}(k) = \begin{cases} w(i) & \text{if } 0 \leq i \leq n-1 \\ \sqcup & \text{otherwise.} \end{cases}$$

$\triangleright$ A Turing machine T with alphabet Σ **accepts** w if a finite number of applications of the map induced by T on $(\widetilde{w}, q_0, 0)$ eventually reaches a configuration of the form $(\star, q_H, \cdot)$.

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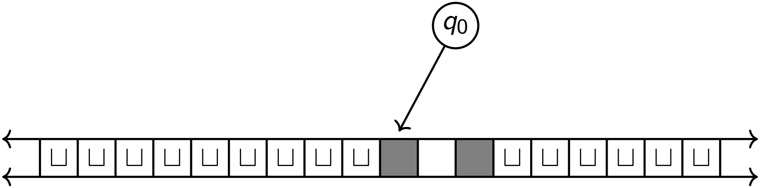
▷ If T does not accept w, we say it **loops** on w.

Example:

- $\Sigma = \{\square, \blacksquare\}$.
- $w = \blacksquare\square\blacksquare$.



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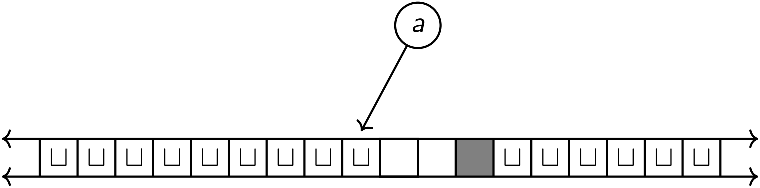

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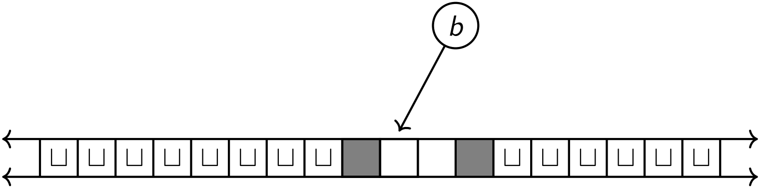

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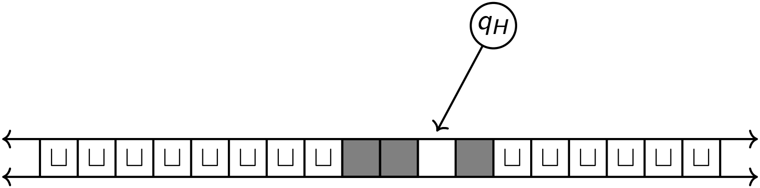

The machine accepts w

Turing machines

Let $L \subset \Sigma^*$ be a language.

- We say L is **recursively enumerable (RE)**:
  if there's a Turing machine T such that $w \in L$ if and only if w is accepted by T.
- We say L is **co-recursively enumerable (co-RE)**:
  if $\Sigma^* \setminus L$ is recursively enumerable.
- We say L is **decidable**:
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Examples

- The language of words in $\{0, 1\}^*$ which represent numbers which are divisible by 7 is decidable.
- The language of words in $\{a, b\}^*$ that are palindromes is decidable

Encoding stuff in languages

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We can talk about decidability of sets of a certain object through their encodings as words in a language.

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(Finitely generated) group

Effective Γ- action

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For a word $w = w_0 w_1 \dots w_{n-1} \in \{0,1\}^n$ consider the cylinder set

$$[w] = \{x \in \{0,1\}^{\mathbb{N}} : x|_{\{0,\dots,n-1\}} = w\}.$$

Effectively closed set

A set $X \subset \{0,1\}^{\mathbb{N}}$ is called **effectively closed** if it is closed and there is a recursively enumerable language $L \subset \{0,1\}^*$ such that

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Intuition: When given $x \in \{0,1\}^{\mathbb{N}}$, there is an algorithm, which if left to work for an arbitrary long time, will eventually tell you if $x \notin X$ (will say nothing if $x \in X$).

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Note: We can replace (for convenience) $\{0,1\}$ with any finite alphabet A and the definition is the same.

Effective dynamics

$\Gamma \curvearrowright X$ can be described by a Turing machine

Let Γ be finitely generated by a symmetric set $S \ni 1_\Gamma$ and $X \subset \{0,1\}^{\mathbb{N}}$ be effectively closed. Given $\Gamma \curvearrowright X$ consider the set

$$Y = \{y \in (\{0,1\}^S)^{\mathbb{N}} : \pi_s(y) = s \cdot \pi_{1_\Gamma}(y) \in X \text{ for every } s \in S\}.$$

Where $\pi_s(y) \in \{0,1\}^{\mathbb{N}}$ is such that $\pi_s(y)(n) = y(n)(s)$.

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Effectively closed action

An action $\Gamma \curvearrowright X$ is effectively closed if Y is an effectively closed set.

Intuition: there is an algorithm telling me (1) when $x \notin X$ and (2) when $x \neq s \cdot y$.

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Here's an equivalent definition:

Effectively closed action v2.0

An action $\Gamma \curvearrowright X$ is effectively closed if X is effectively closed and there is a Turing machine T, which, given $s \in S$, $n \in \mathbb{N}$ and "oracle" access to all coordinates of $x \in X$, can compute the value $(sx)(n)$.

Examples

Odometer

$\mathbb{Z} \curvearrowright (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ given by $x \mapsto x + 1$ in binary.

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Subshifts of finite type

SFTs are topologically conjugate to effectively closed actions.

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$\mathbb{Z} \curvearrowright (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ given by $x \mapsto x+1$ in binary.

Subshifts of finite type

SFTs are topologically conjugate to effectively closed actions.

Topological factors

$\Gamma \curvearrowright Y$ is a topological factor of $\Gamma \curvearrowright X$ if there exists a continuous surjective map $\varphi \colon X \to Y$ which is Γ-equivariant ($g\phi(x) = \phi(gx)$ for every $g \in \Gamma, x, y \in X$).

Topological factors of effectively closed actions are effectively closed.

Examples

Consider $X = \{0,1\}^{\mathbb{N}}$ and let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be non-empty words in $\{0,1\}^*$ such that

$$X = [u_1] \sqcup [u_2] \sqcup \cdots \sqcup [u_n] = [v_1] \sqcup [v_2] \sqcup \cdots \sqcup [v_n].$$

Let φ be the homeomorphism of $\{0,1\}^{\mathbb{N}}$ which maps every cylinder $[u_i]$ to $[v_i]$ by replacing prefixes, that is

$$\varphi(u_i x) = v_i x \text{ for every } x \in \{0,1\}^{\mathbb{N}}.$$

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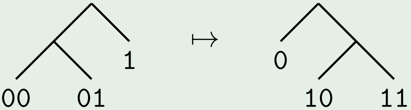
$u_1 = 00, u_2 = 01, u_3 = 1$ and $v_1 = 0, v_2 = 10, v_3 = 11$.

$$\varphi(0101010\ldots) = 1001010\ldots \quad \varphi(0000000\ldots) = 0000000\ldots$$
$$\varphi(1111111\ldots) = 1111111\ldots \quad \varphi(0011001\ldots) = 011001\ldots$$



Natural action of Thompson's groups

- F is the group of all such homeomorphisms where $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are given in lexicographical order.
- T is the group of all such homeomorphisms where $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are given in lexicographical order up to a cyclic permutation.
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Their natural action on $\{0,1\}^{\mathbb{N}}$ is effectively closed.

Why care about effective actions?

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(1) Many interesting classes of dynamical systems are effective.

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(2) Many problems about those classes can be answered in terms of computability.

Universality

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This is easy to prove for Turing machines, and from there the result takes different shapes in different contexts:

- There is a **universal** finitely presented group which contains copies of all recursively presented groups.
- There is a **universal** polynomial which can realize all diophantine sets by fixing one of its variables as some integer.

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Let us see a dynamical version of this notion of simulation.

Subshift of finite type

Let A be a finite set and consider $A^{\mathbb{Z}^d} = \{x \colon \mathbb{Z}^d \to A\}$ with the prodiscrete topology and the action $\mathbb{Z}^d \curvearrowright A^{\mathbb{Z}^d}$ given by

$$(nx)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

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Subshift of finite type

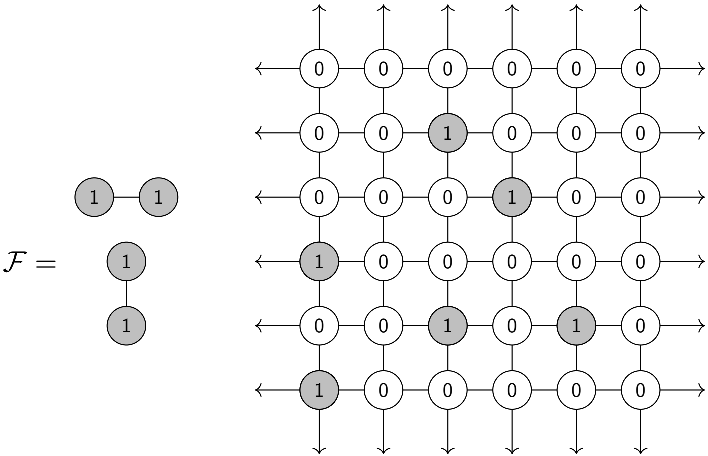
A set $Z \subseteq A^{\mathbb{Z}^d}$ is a $\mathbb{Z}^d$-**subshift of finite type** (SFT) is there is a finite set $F \subseteq \mathbb{Z}^d$ and $\mathcal{F} \subseteq A^F$ such that $z \in Z$ if and only if

$$(mz)|_F \notin \mathcal{F} \text{ for every } m \in \mathbb{Z}^d.$$

Intuition: A subshift is of finite type if it is the set of configurations $x \in A^{\mathbb{Z}^d}$ which avoid a finite list of forbidden patterns (represented by $\mathcal{F}$).

Examples

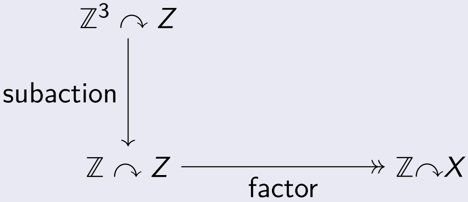
Hard-square shift. $Z = \{x : \mathbb{Z}^2 \to \{0,1\}\}$ such that there are no vertical or horizontally adjacent 1s.



What results are known?

Hochman's theorem, 2009

Every effectively closed action $\mathbb{Z} \curvearrowright X$ is the topological factor of a subaction of a $\mathbb{Z}^3$-subshift of finite Z.



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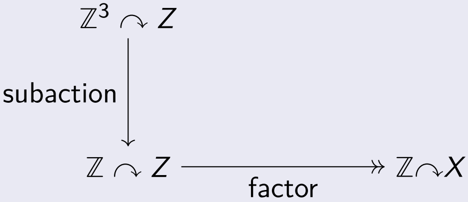

Moreover, the factor is nice (mod a group rotation, 1-1 in a set of full measure with respect to any invariant measure.)

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$$\begin{array}{ccc} \mathbb{Z}^3 \curvearrowright Z & & \\ \text{subaction} \Big\downarrow & & \\ \mathbb{Z} \curvearrowright Z & \xrightarrow[\text{factor}]{\hspace{4em}} & \mathbb{Z} \curvearrowright X \end{array}$$

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An action $\Gamma \curvearrowright X$ on a metric space is expansive if there is $C > 0$ such that whenever $d(gx, gy) \leq C$ for every $g \in \Gamma$ then $x = y$.

$$\text{expansive} + \text{zero-dimensional} \iff \text{subshift.}$$

Expansive effectively closed actions $\mathbb{Z} \curvearrowright X$ are topologically conjugate to effectively closed subshifts.

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Effectively closed subshift

A set $Z \subseteq A^{\mathbb{Z}}$ is an **effectively closed subshift** is there is a recursively enumerable set $\mathcal{F}$ of words in A^* such that $z \in Z$ if and only if

$$(mz)|_{\{0,\dots,n-1\}} \notin \mathcal{F} \text{ for every } m \in \mathbb{Z} \text{ and } n \in \mathbb{N}.$$

Aubrun-Sablik 2013, Durand-Romaschenko-Shen 2010

Every effectively closed expansive action $\mathbb{Z} \curvearrowright X$ is topologically conjugate to the $\mathbb{Z}$-subaction of a symbolic factor of a $\mathbb{Z}^2$-SFT Z.

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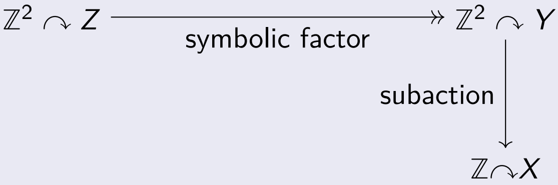

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- Existence of $\mathbb{Z}^2$-SFTs where the action is free (Berger, Robinson)
- Undecidability of whether a $\mathbb{Z}^2$-SFT X given by a finite list of forbidden patterns is empty (Berger)
- Characterization of the topological entropies of $\mathbb{Z}^2$-SFTs (Hochman-Meyerovitch).

- Let $X \subset A^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$-subshift.
- Let $B_n = [\![0, n-1]\!]^d$.
- Let $L_n(X) = \{p \in A^{B_n} : \text{ there is } x \in X, x|_{B_n} = p\}$.

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Topological entropy

The topological entropy of X is given by the formula

$$h(\mathbb{Z}^d \curvearrowright X) = \lim_{n\to\infty} \frac{1}{|B_n|} \log(|L_n(X)|) = \inf_{n\to\infty} \frac{1}{|B_n|} \log(|L_n(X)|).$$

The case $d = 1$

Example

Let $X \subset \{0,1\}^{\mathbb{Z}}$ be the subshift of finite type where no pair of 1s can be adjacent. It is easy to verify that

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There are countably many SFTs. What is the class of their topological entropies?

The case $d = 1$

D. Lind 1986

The entropies of $\mathbb{Z}$-subshifts of finite type are precisely the non-negative rational multiples of logarithms of Perron numbers λ

$$h(\mathbb{Z} \curvearrowright X) = \frac{p}{q} \log(\lambda).$$

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What about $\mathbb{Z}^d$ for $d \geq 2$?

The case $d \geq 2$

M. Hochman and T. Meyerovitch 2010

The entropies of $\mathbb{Z}^d$-subshifts of finite type for $d \geq 2$ are precisely the non-negative real numbers which are **upper semi-computable**

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A real r is **upper semi-computable** if there is a Turing machine which on input $n \in \mathbb{N}$ outputs a rational $q_n \in \mathbb{Q}$ such that

$$\inf_{n \in \mathbb{N}} q_n = r$$

Proof sketch

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- Let $L_k^{\text{loc,n}}(X)$ be the set of patterns $p \in A^{B_k}$ for which there is a pattern $q \in A^{B_n}$ such that $p = q|_{B_k}$ and q contains no forbidden patterns.

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Proof sketch

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$$\mathbb{Z}^2 \curvearrowright X \xrightarrow[\text{symbolic factor}]{} \mathbb{Z}^2 \curvearrowright Y$$

SFT

$h(\mathbb{Z}^2 \curvearrowright X) = 0$

subaction: $\mathbb{Z}^2 \curvearrowright Y \longrightarrow \mathbb{Z} \curvearrowright Z$

Proof sketch

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$\mathbb{Z}^2 \curvearrowright X \twoheadrightarrow \mathbb{Z}^2 \curvearrowright Y$

symbolic factor

SFT

$h(\mathbb{Z}^2 \curvearrowright X) = 0$

subaction

$\mathbb{Z} \curvearrowright Z$

Let $X \subset A^{\mathbb{Z}^2}$ as above, and consider

$$X' \subseteq X \times \{0, 1, \ldots, \kappa\}^{\mathbb{Z}^2}$$

where $x' = (x, t) \in X'$ satisfies that for every k:

$$\phi(x)(k) = 0 \iff t(k) = 0.$$

Proof sketch

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$\mathbb{Z}^2 \curvearrowright X$ —— symbolic factor ——↠ $\mathbb{Z}^2 \curvearrowright Y$

SFT

$h(\mathbb{Z}^2 \curvearrowright X) = 0$

$\mathbb{Z}^2 \curvearrowright Y$ —— subaction ——→ $\mathbb{Z} \curvearrowright Z$

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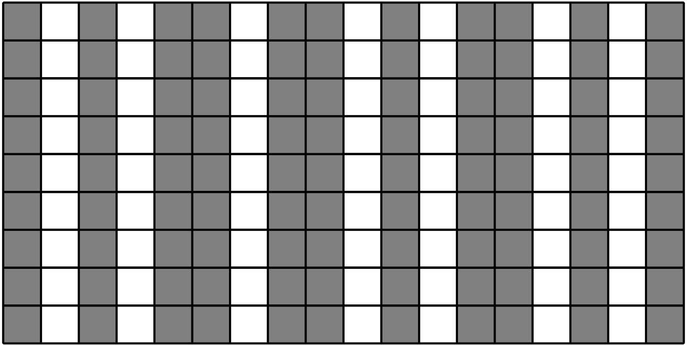
Intuition: We create κ independent copies of every symbol that maps into 1 to generate entropy with density $r' \log(\kappa)$.



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0	1	0	3	0	0	2	0	0	1	0	3	0	0	2	0	3	0

$$|L_n(X')| \approx |L_n(X)| \cdot \kappa^{n^2 q_n}$$

0	1	0	3	0	0	2	0	0	2	0	2	0	0	3	0	1	0
0	2	0	3	0	0	2	0	0	1	0	2	0	0	2	0	2	0
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0	2	0	3	0	0	1	0	0	1	0	1	0	0	3	0	3	0
0	1	0	3	0	0	2	0	0	1	0	3	0	0	2	0	3	0

$$|L_n(X')| \approx |L_n(X)| \cdot \kappa^{n^2 q_n}$$

As $q_n \to r$ and $\log |L_n(X)| = o(n^2)$, it follows that

$$h(\mathbb{Z}^2 \curvearrowright X') = r' \log(\kappa) = r.$$

Wrapping up

- Many well-known dynamical systems are effective.
- Several problems in dynamics admit solutions in terms of computability.
- Universality results can be used as black boxes to solve problems.

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Next week

- A strong universality property for certain classes of non-amenable groups.
- **Self-simulable groups** (effective actions are factors of SFTs)
- Rigidity properties of these groups.
- A computability characterization of the (?) amenability of Thompson's F.

Thank you for your attention!

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