The Dobrushin Lanford Ruelle theorem on steroids

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Joint with Ricardo **Gómez Aíza**, Brian **Marcus** and Siamak **Taati**

University of British Columbia

Probability Seminar
Vancouver
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The DLR theorem

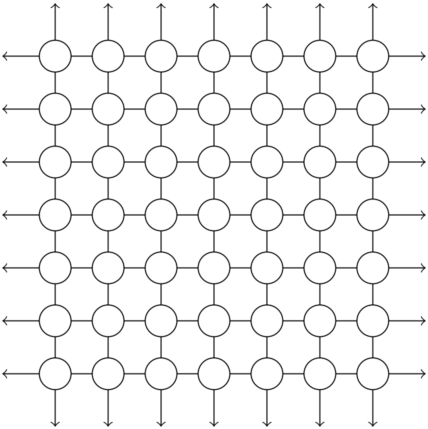
Theorem

Let Σ be a finite set of symbols. Let $X \subseteq \Sigma^{\mathbb{Z}^d}$ be a d-dimensional subshift, Φ an absolutely summable interaction on X, and f_Φ an associated energy observable.

1. (Dobrushin theorem)
   Assume that X is D-mixing. Then, every shift-invariant Gibbs measure for Φ is an equilibrium measure for Φ.
2. (Lanford–Ruelle theorem)
   Assume that X is a subshift of finite type (SFT). Then, every equilibrium measure for Φ is a Gibbs measure for Φ.

Example of Relative System

Environment Θ



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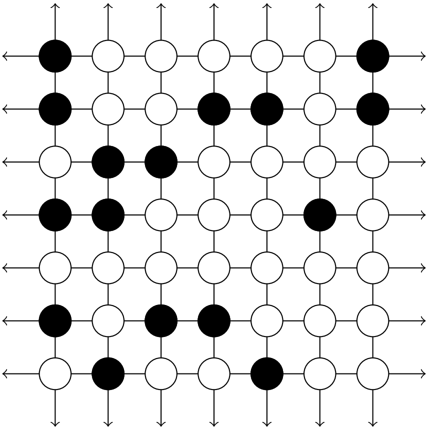

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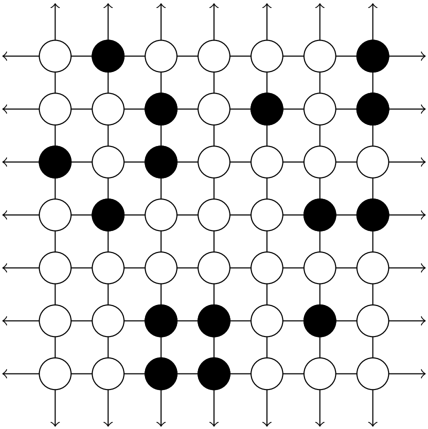

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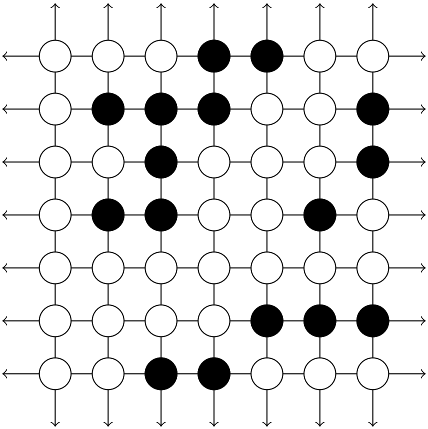

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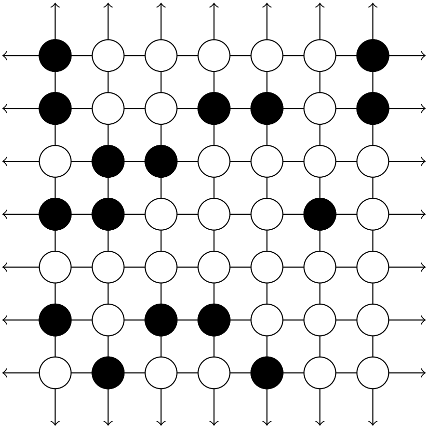

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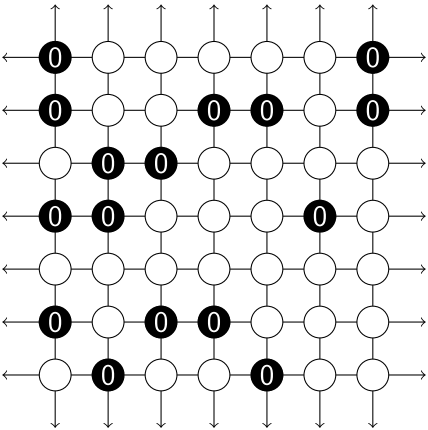

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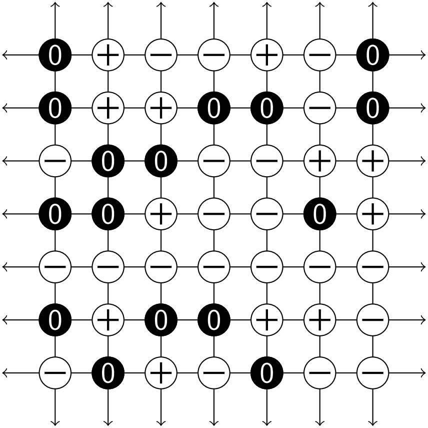

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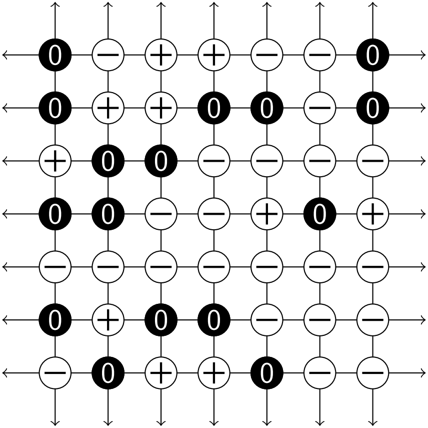

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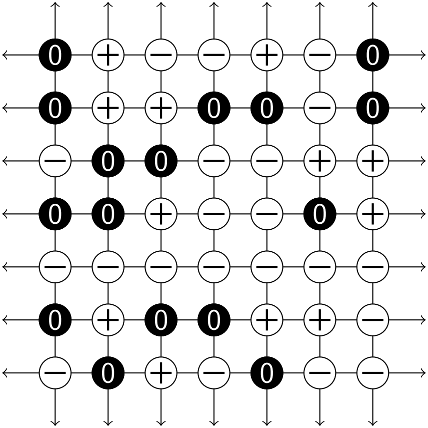

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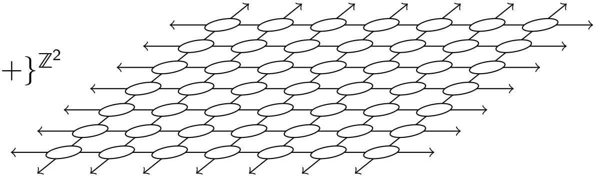
Example of Relative System (continued)

$X_\theta \subseteq \{0, -, +\}^{\mathbb{Z}^2}$



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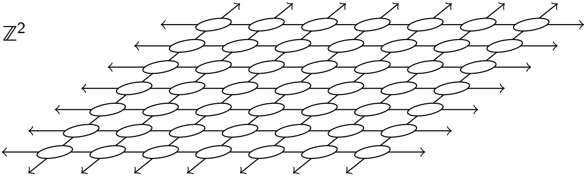

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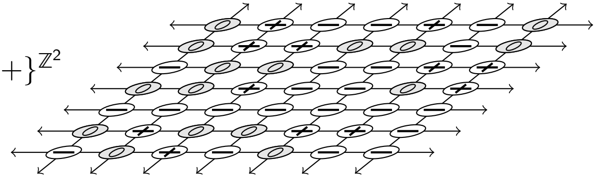

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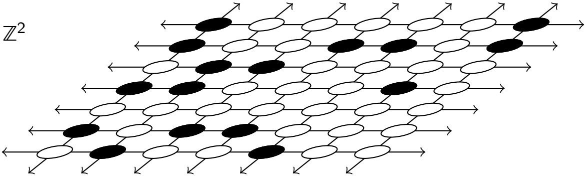

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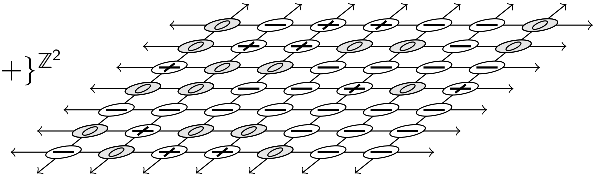

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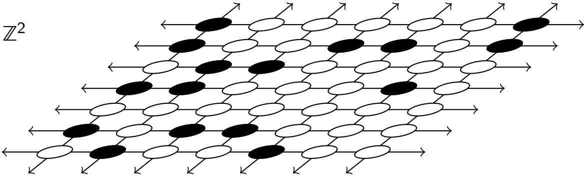

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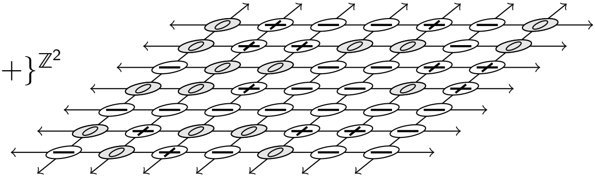

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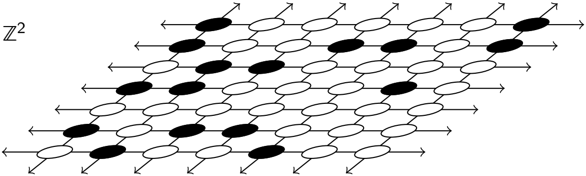

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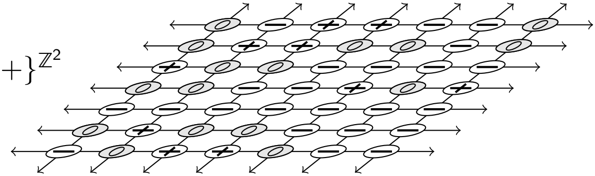

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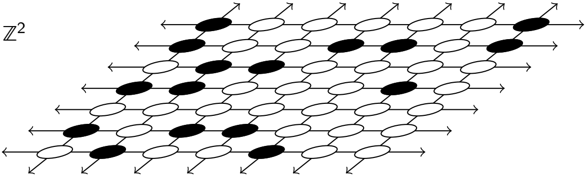

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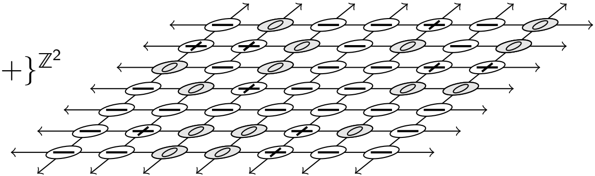

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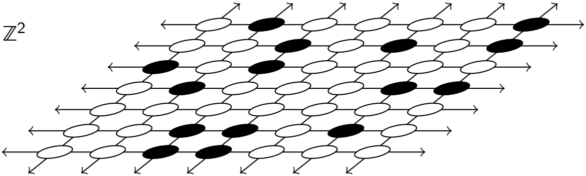

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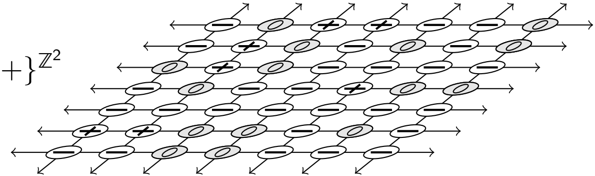

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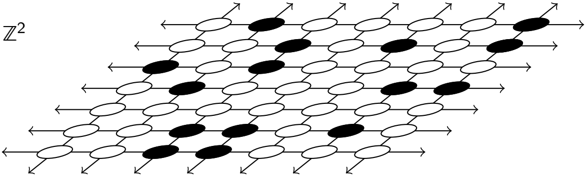

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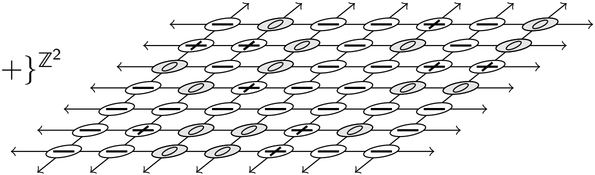

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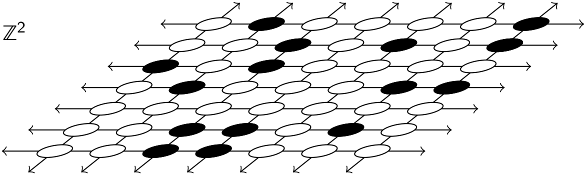

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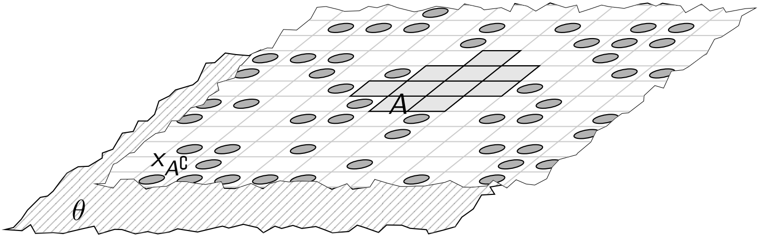

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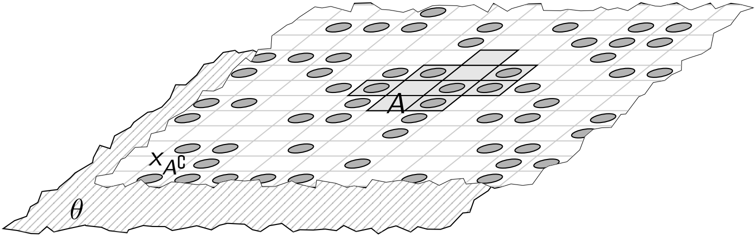

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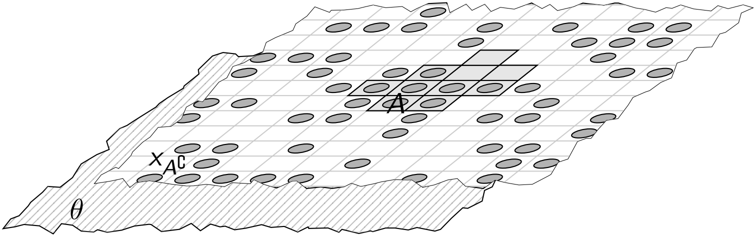

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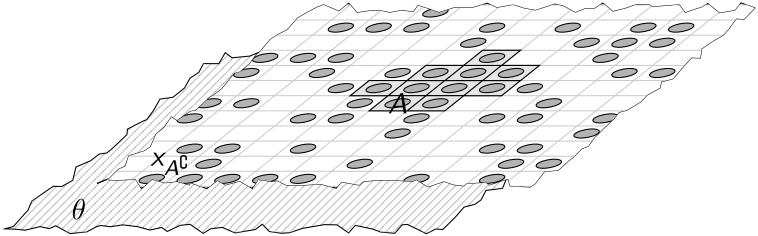

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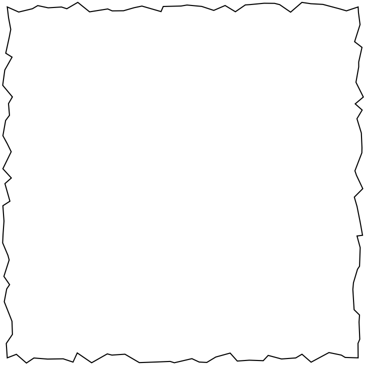

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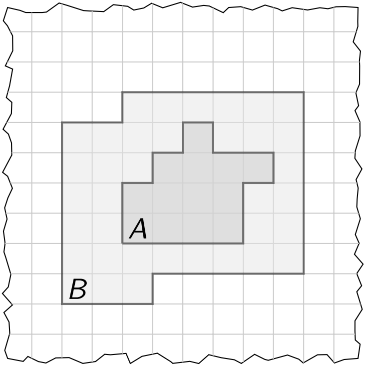

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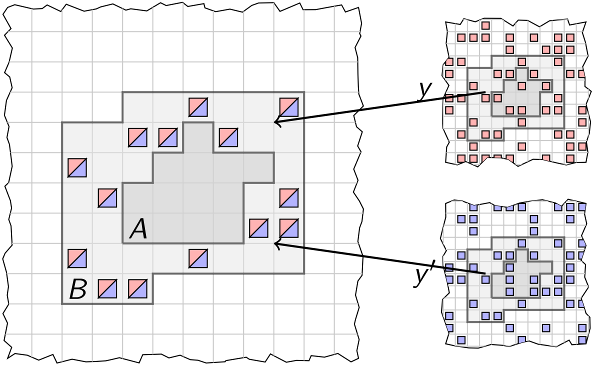

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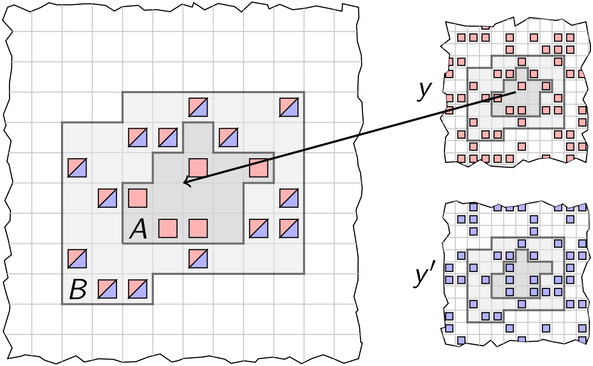

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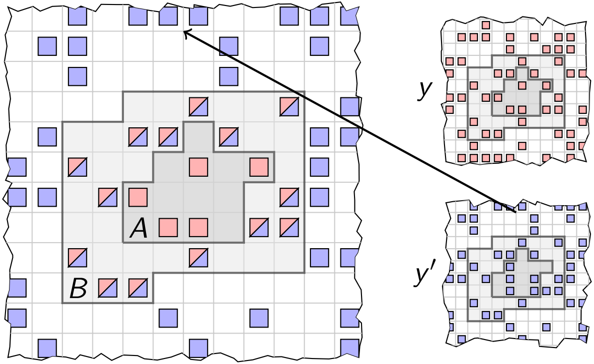

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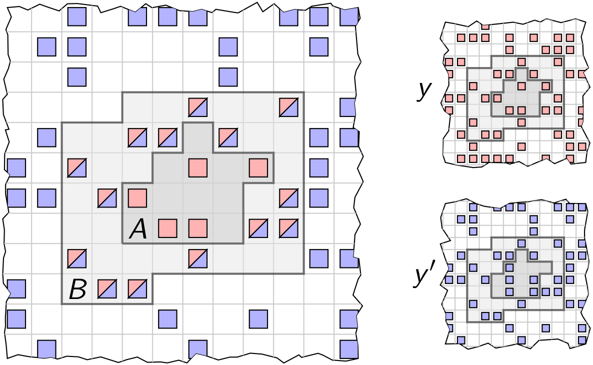

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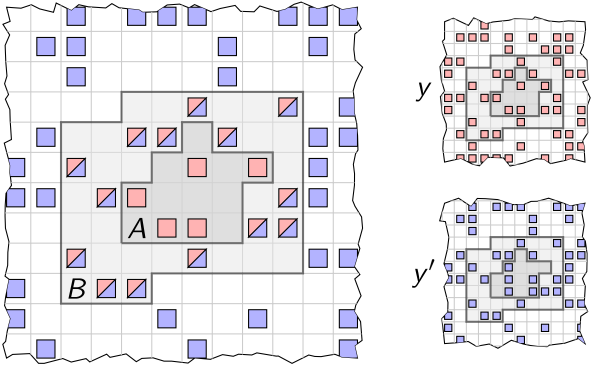
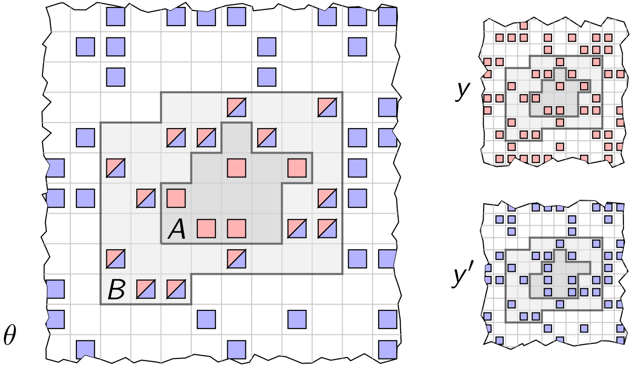


Remark: for subshifts, weak TMP is much weaker than SFT.

relative weak topological Markov property (relative weak TMP)

Given a relative system Ω and a G-invariant measure ν on environment Θ, Ω satisfies the *relative weak TMP*, relative to ν, if for each finite $A \subset G$, there is a finite $B \supseteq A$ such that for ν-a.e. $\theta \in \Theta$, X_θ satisfies weak TMP with B as a "witness" for A.



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- For some finite $A \subset G$,

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- Find a set D of positive density, $d(D)$, such that copies of B centered at elements of D are disjoint.

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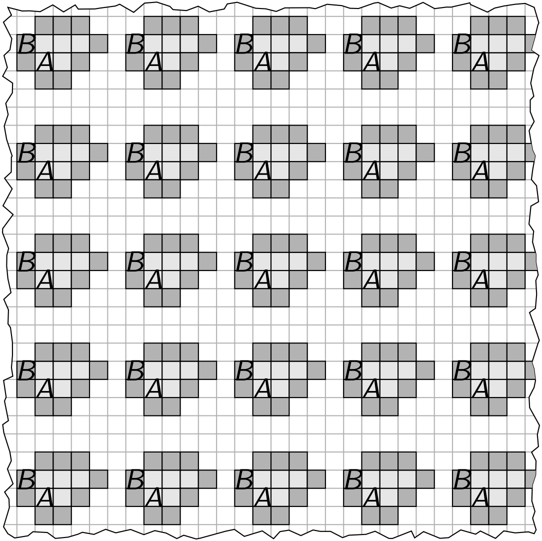

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$$H_{\mu_+}(\xi^{F_n}|\mathcal{F}_\Theta) - H_\mu(\xi^{F_n}|\mathcal{F}_\Theta) \geq \varepsilon d(D)|F_n| + o(|F_n|).$$

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LR: Relative Equilibrium implies Relative Gibbs

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Assume that X is a G-subshift that satisfies weak TMP. Then any measure of maximal pressure relative to ν, in LW sense, is Gibbs relative to ν.

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Proof: Show that if X satisfies weak TMP, then Ω satisfies relative weak TMP. Apply relative LR Theorem. $\square$

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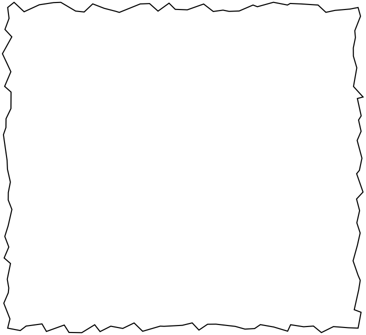

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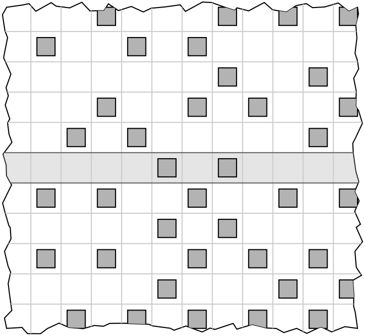

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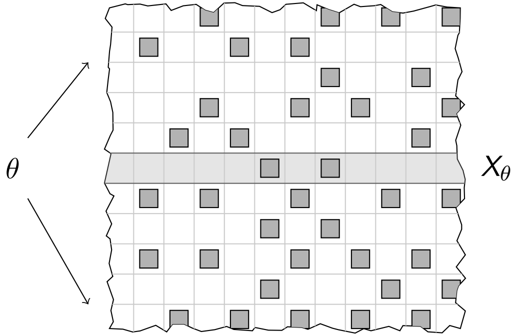

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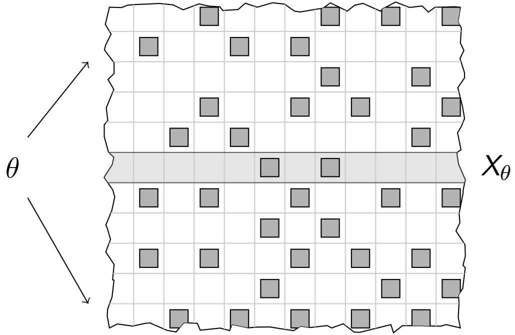

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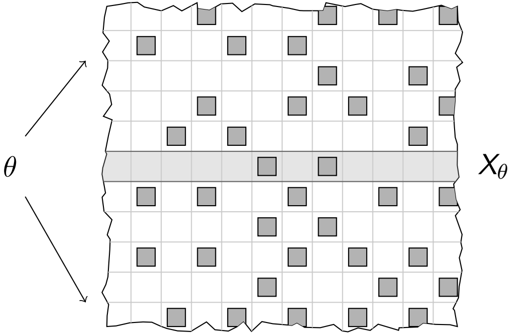

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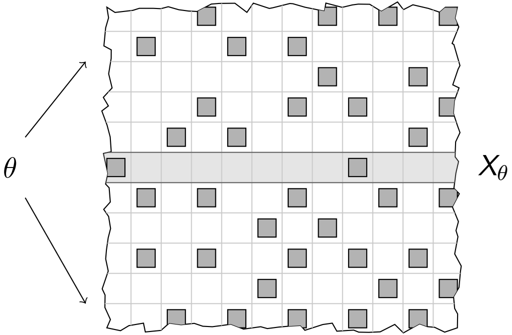

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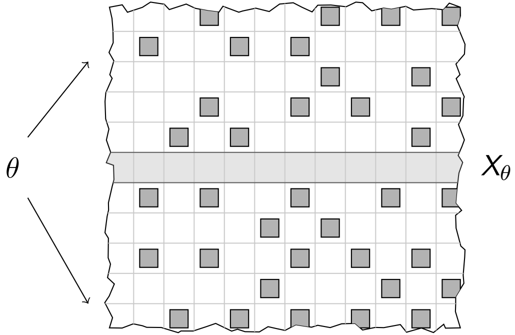

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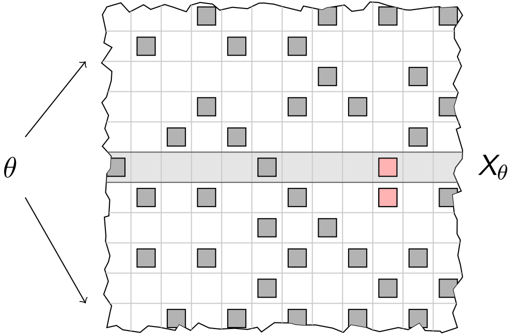

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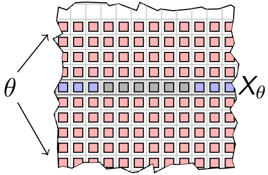

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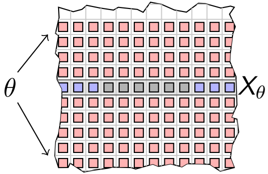

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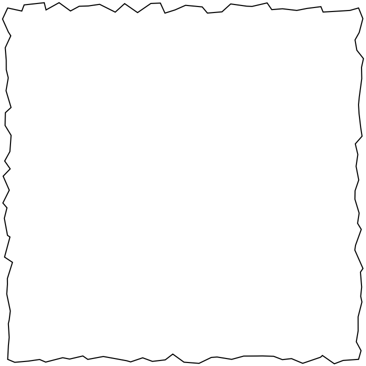

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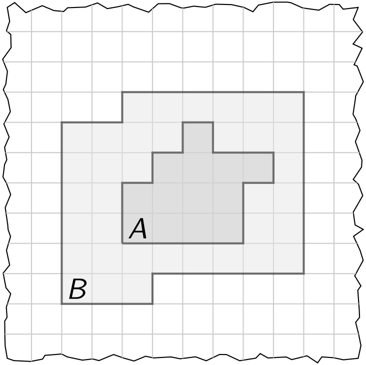

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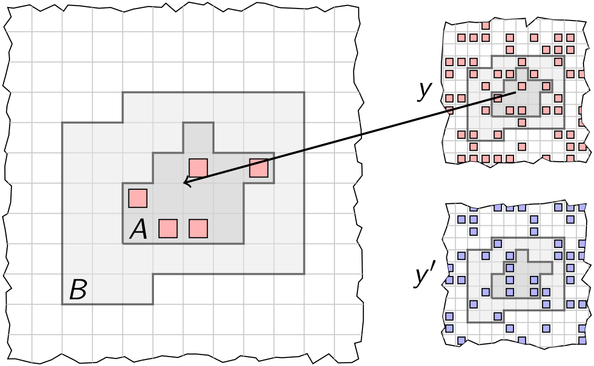

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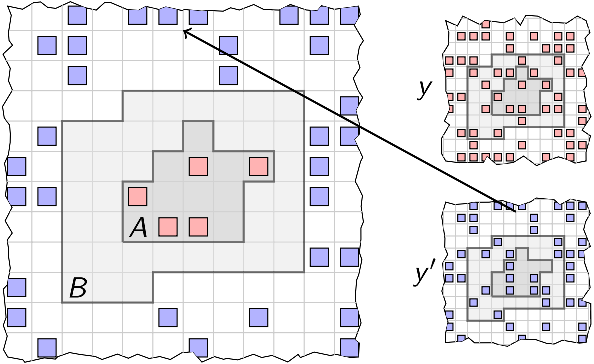

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