The domino problem for self-similar structures

Sebastián Barbieri and Mathieu Sablik

LIP, ENS de Lyon – CNRS – INRIA – UCBL – Université de Lyon

Aix-Marseille Université

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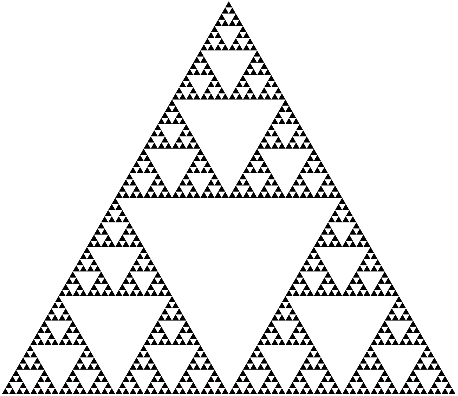
Tilings fractals



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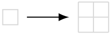
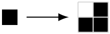
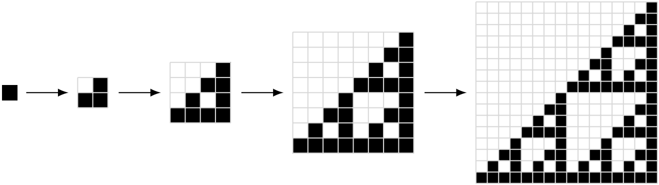
 and 



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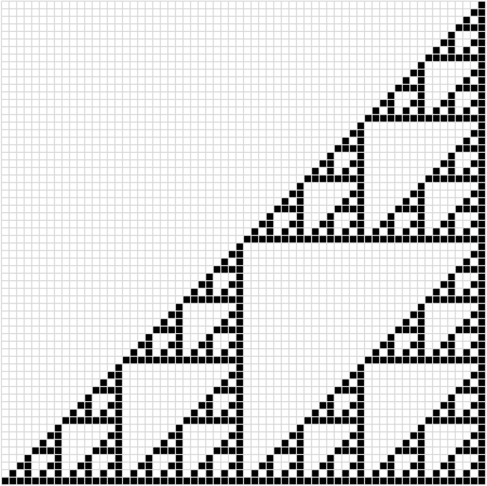

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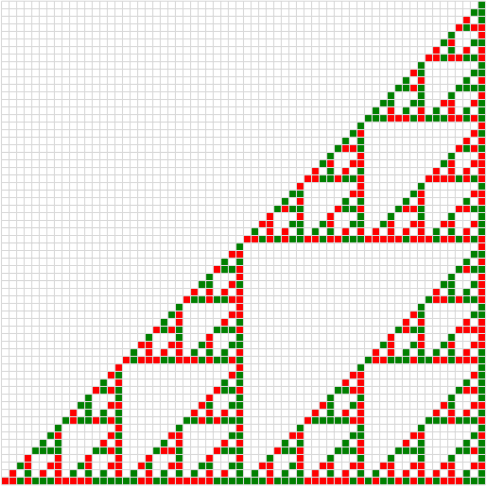

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Goals of this talk

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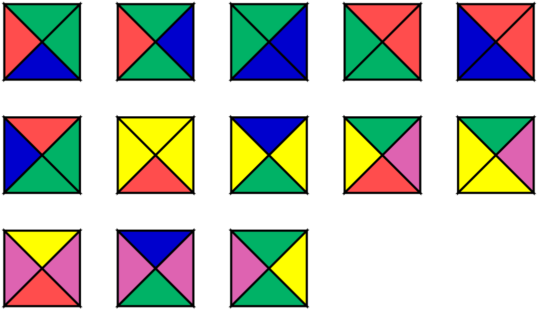
- Constructing a framework to study tilings over fractals.
- In particular tilings with a finite number of local constrains.
- Decidability aspects : The domino problem.

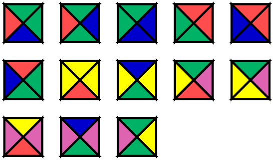
The domino problem

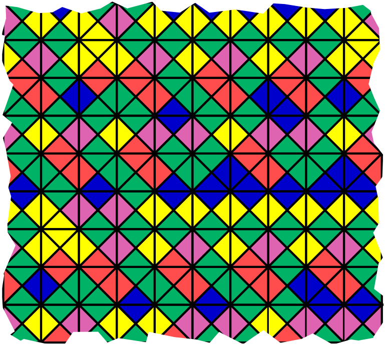
Wang tiles are unit squares with colored edges.



The domino problem

Goal : cover the plane with squares in such a way that matching edges have the same color.





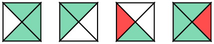
The domino problem

The domino problem

Is there a Turing machine which given on entry a set of Wang tiles decides whether they tile the plane or not ?

What about periodicity in Wang Tilings ?

Let τ be a finite set of Wang tiles.



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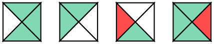
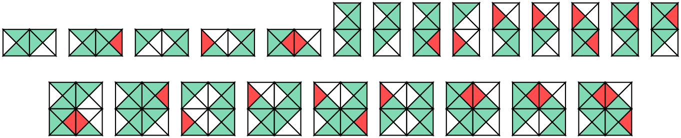


It is easy to generate all the local patterns which satisfy the rules.



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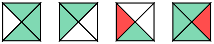

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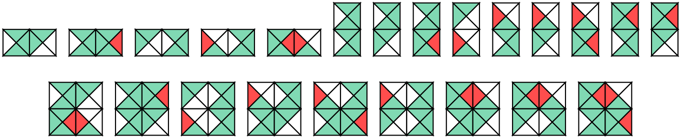
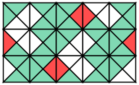
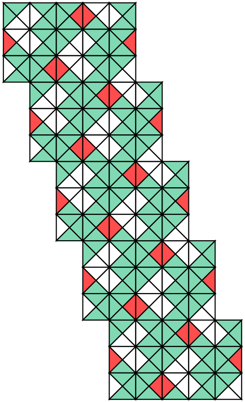


If you find a locally admissible pattern with *matching edges*, then τ tiles the plane periodically.



Tiling of the plane using periodic patch



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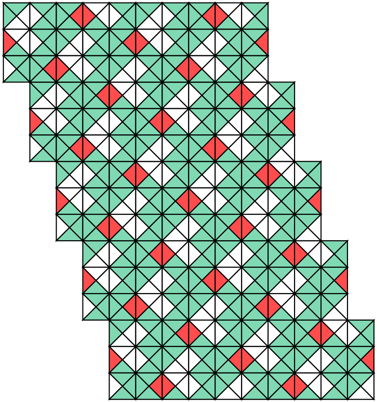

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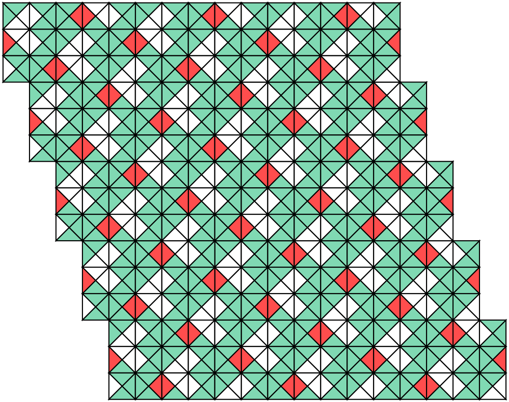

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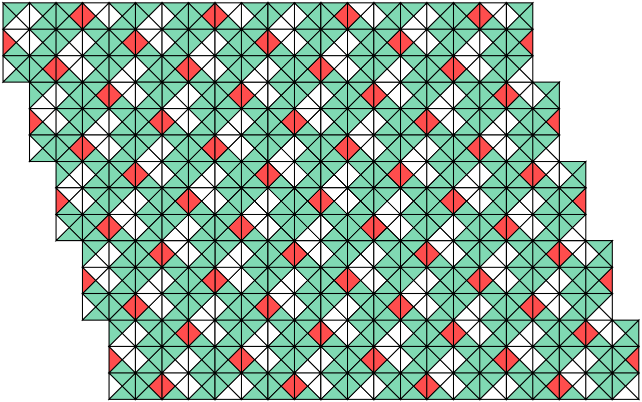

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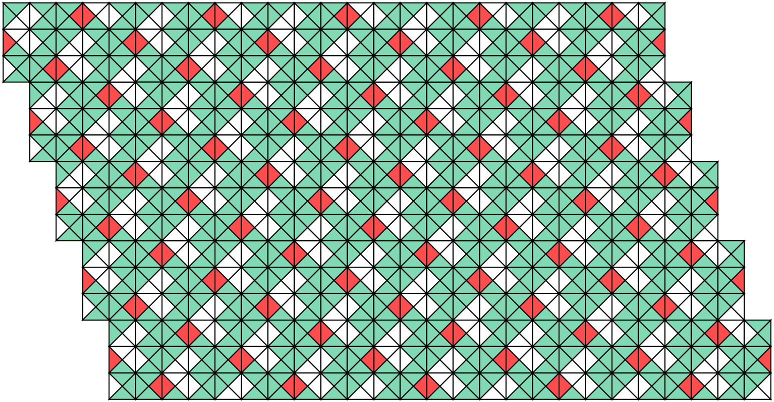

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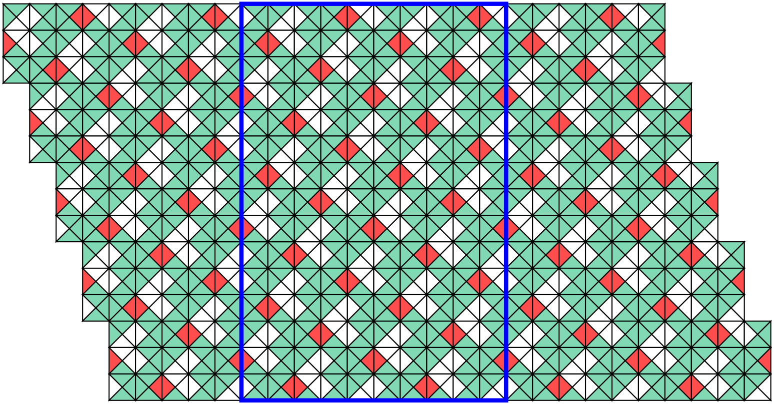

Wang's conjecture

Wang's conjecture (1961)

If a set of Wang tiles can tile the plane, then they can always be arranged to do so periodically.

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If a set of Wang tiles can tile the plane, then they can always be arranged to do so periodically.

If Wang's conjecture is true, we can decide if a set of Wang tiles can tile the plane !

Semi-algorithm 1 :

1. Accept if there is a periodic configuration.
2. loops otherwise

Semi-algorithm 2 :

1. Accept if a block $[0, n]^2$ cannot be tiled without breaking local rules.
2. loops otherwise

Wang's conjecture

Theorem[Berger 1966]

Wang's conjecture is FALSE

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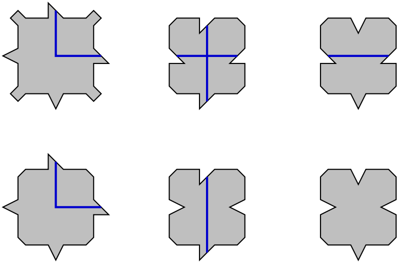
Wang's conjecture is FALSE

His construction encodes a Turing machine using an alphabet of size 20426.

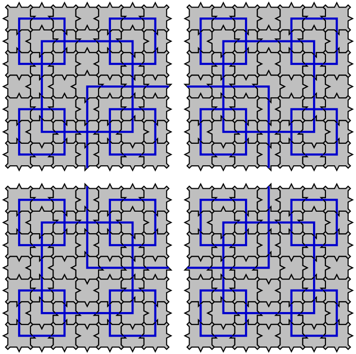
His proof was later simplified by Robinson[1971]. A proof with a different approach was also presented by Kari[1996].

Robinson tileset

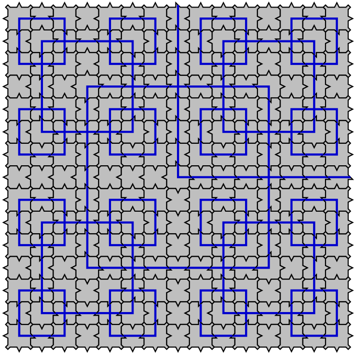
The Robinson tileset, where tiles can be rotated.



From macro-tiles of level n to macro-tiles of level $n + 1$



$\Rightarrow$



The easy case : A line.

Theorem :

The set of colorings of a line subject to a finite number of patterns not appearing can be characterized as the set of bi-infinite walks in a finite graph.

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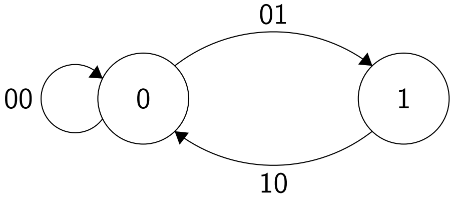
Example : Consider the set of words $X \subset \{0,1\}^{\mathbb{Z}}$ where $\{11\}$ does not appear.



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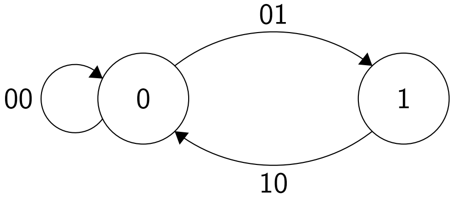

The domino problem is decidable in the line.

Our work

So far we have :

- $\mathtt{DP}(\mathbb{Z})$ is decidable.
- $\mathtt{DP}(\mathbb{Z}^2)$ is undecidable.

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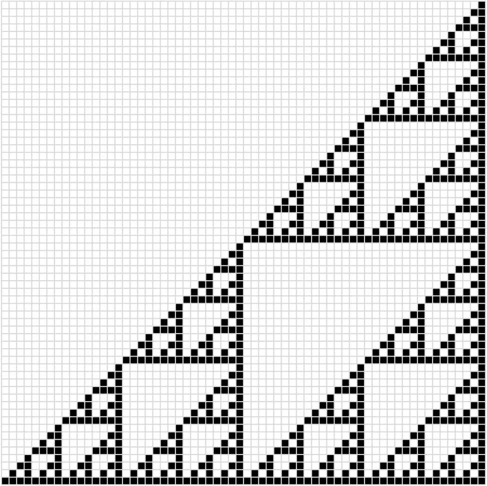
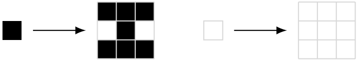
What about intermediate structures ?

Toy case : Sierpiński triangle

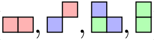


Setting

We fix a two-dimensional substitution s over the alphabet $\mathcal{A} = \{□, ■\}$ such that □ gets sent to a rectangle of □ and ■ to a mixture of both.

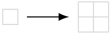
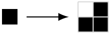


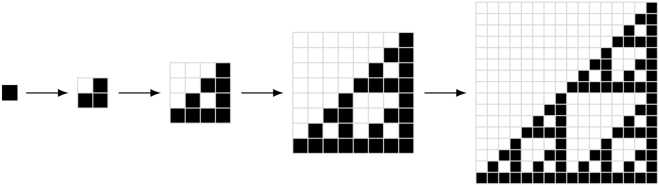
The input of the problem is a finite alphabet ex : $\Sigma = \{□, □, □\}$ and a finite set of forbidden patterns, ex :

$\mathcal{F} = \{$  $\}.$

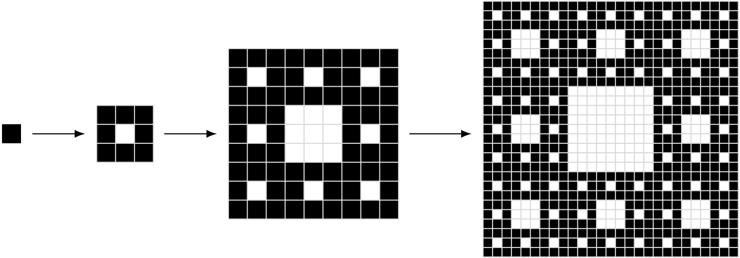
Example 1 : Sierpiński triangle

Consider the alphabet $\mathcal{A} = \{\square, \blacksquare\}$ and the self-similar substitution s such that :

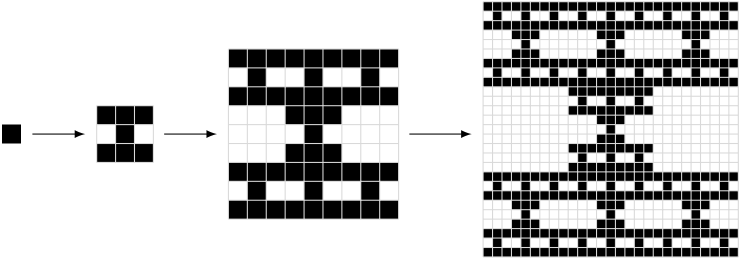
 and 



Example 2 : Sierpiński carpet



Example 3 : The Bridge.



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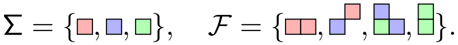

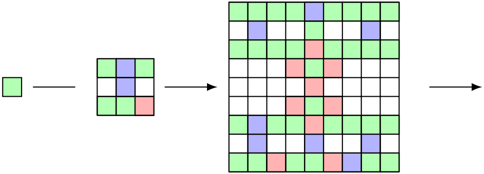

$$\mathtt{DP}(s) = \{\langle \Sigma, \mathcal{F} \rangle \mid s \text{ can be tiled by } \Sigma, \mathcal{F}\}.$$

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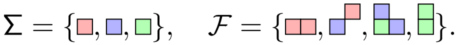

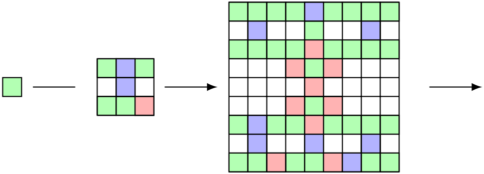

$$\mathrm{DP}(s) = \{\langle \Sigma, \mathcal{F} \rangle \mid s \text{ can be tiled by } \Sigma, \mathcal{F}\}.$$

Domino problem : for which s is $\mathrm{DP}(s)$ decidable ?

Back to the fractal structures...

Why are we interested in this kind of structures ?

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- They are a nice class of intermediate structures between $\mathbb{Z}$ and $\mathbb{Z}^2$ defined by a $\{0, 1\}$-matrix.
- It is easy to calculate a Hausdorff dimension (in this case box-counting dimension). Is there a threshold in the dimension which enforces undecidability ?
- These objects are in fact **subshifts**. And they can be defined by local rules (sofic subshifts) according to Mozes Theorem.

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Theorem :

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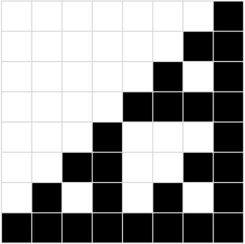

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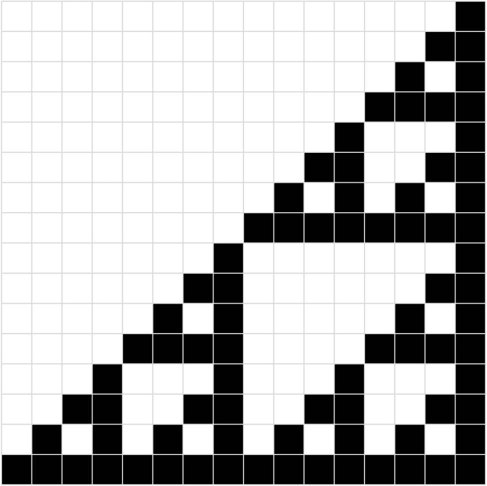

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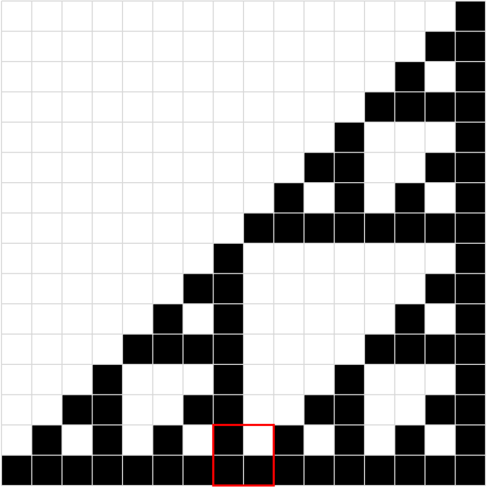

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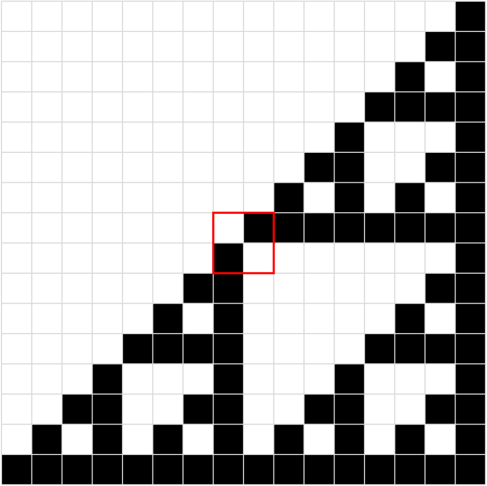

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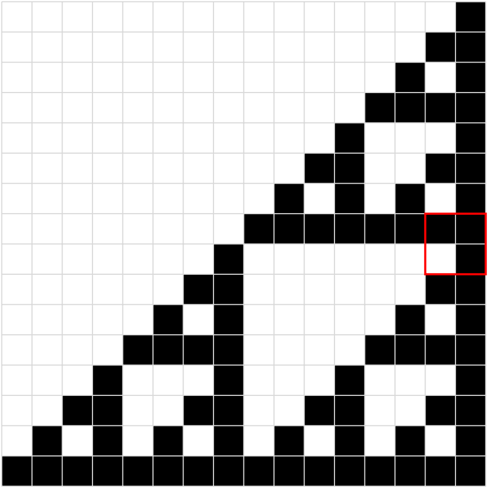

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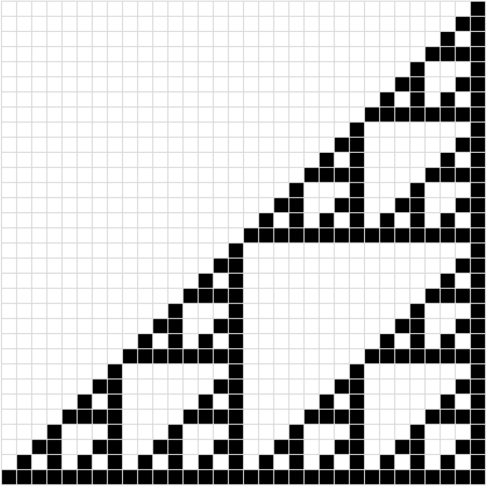

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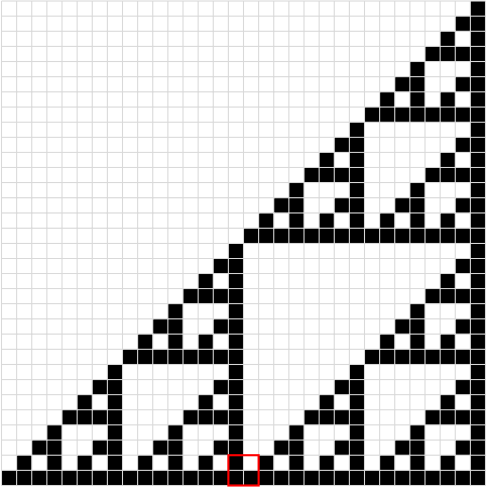

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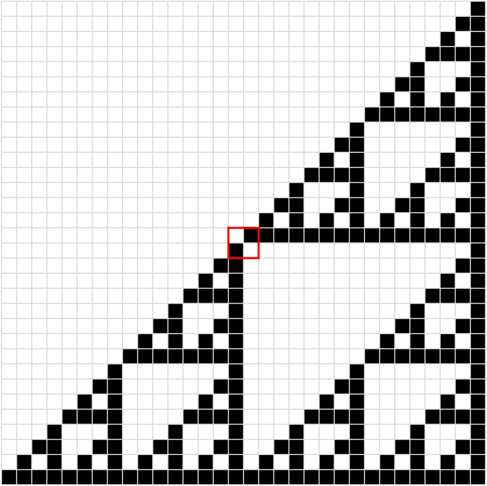

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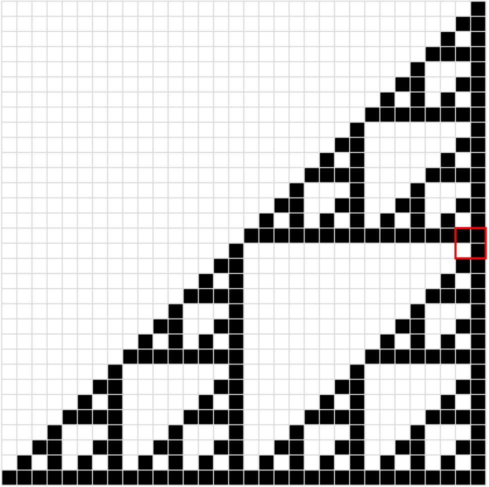

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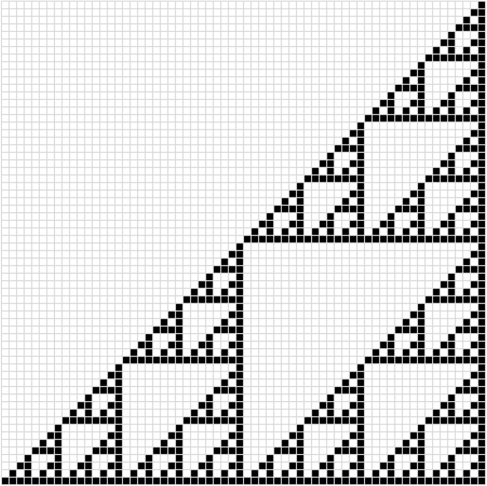

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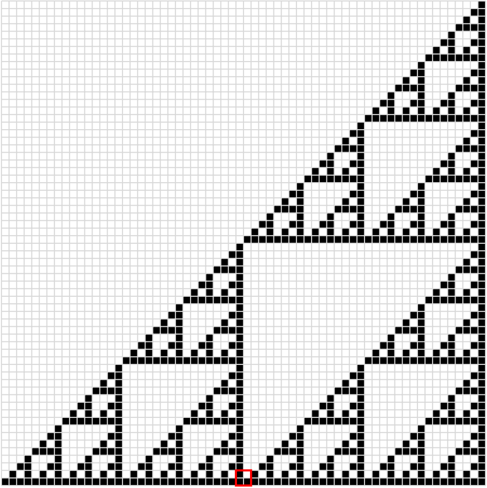

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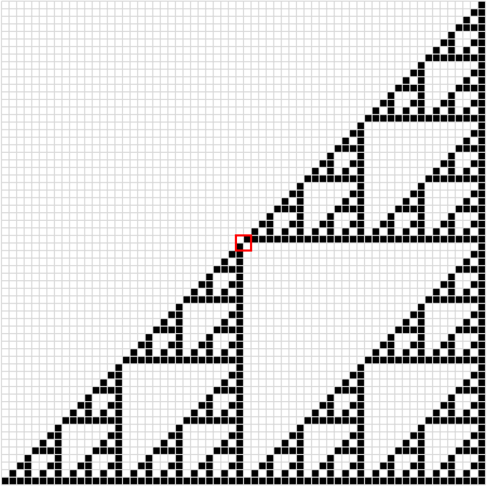

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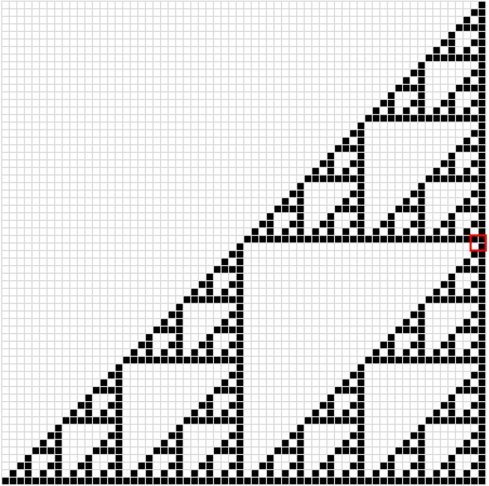

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This technique can be extended to a big class of self-similar substitutions which we call **Bounded connectivity substitutions** !

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Theorem :
The domino problem is undecidable in the Sierpiński carpet.

Proof strategy :

- Suppose we can simulate substitutions over the Sierpiński carpet (using a bigger alphabet).

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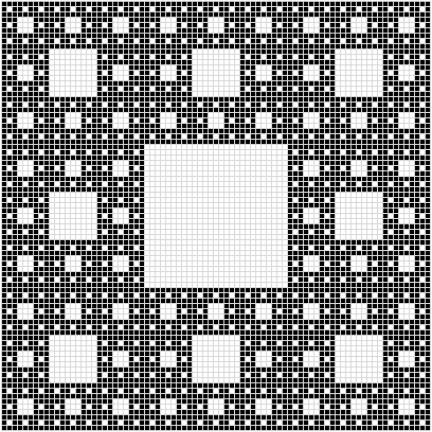
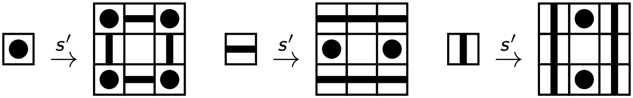


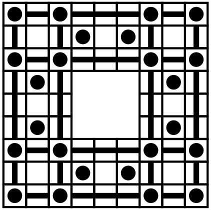
Toy case 2 : Sierpiński carpet.

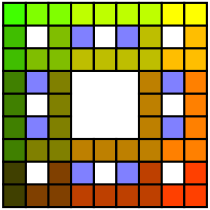
Suppose we can realize the following substitution using local rules.



Toy case 2 : Sierpiński carpet.

$\sqsubset X(\mathcal{F}) \longleftrightarrow$





Toy case 2 : Sierpiński carpet.

Theorem :

The domino problem is undecidable in the Sierpiński carpet.

Proof strategy :

- Suppose we can simulate substitutions over the Sierpiński carpet (using a bigger alphabet).
- Use the substitution shown above to simulate arbitrarily big patterns of a $\mathbb{Z}^2$-tiling (say by Wang tiles)

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  - $\mathtt{DP}(\mathbb{Z}^2)$ is reduced to the domino problem in the carpet.

It only remains to show that we can simulate substitutions with local rules.

Toy case 2 : Sierpiński carpet and Mozes

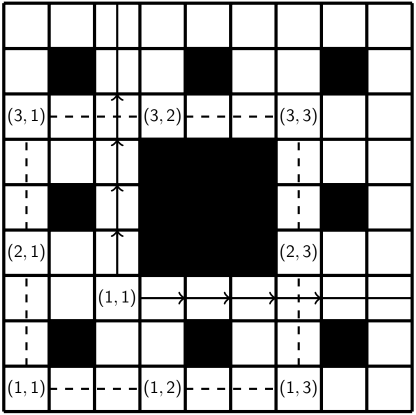
We need to prove a modified version of Mozes' theorem :

Theorem : Mozes.

The subshifts generated by $\mathbb{Z}^2$-substitutions are sofic (are the image of a subshift of finite type under a factor map)

We can prove a similar version in our setting for some substitutions. Among them the Sierpiński carpet.

Toy case 2 : Sierpiński carpet and Mozes



Conclusion

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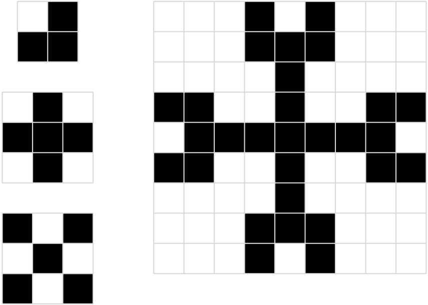
Bounded Connectivity



Decidable domino problem

Conclusion

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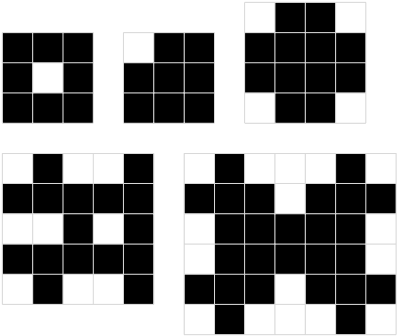
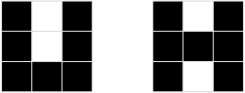
Strong grid

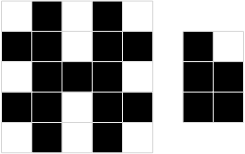


Undecidable domino problem

Conclusion

Isthmus

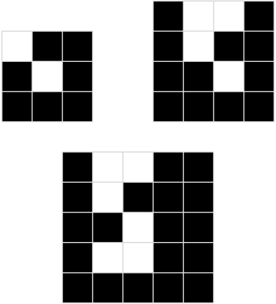




Unknown

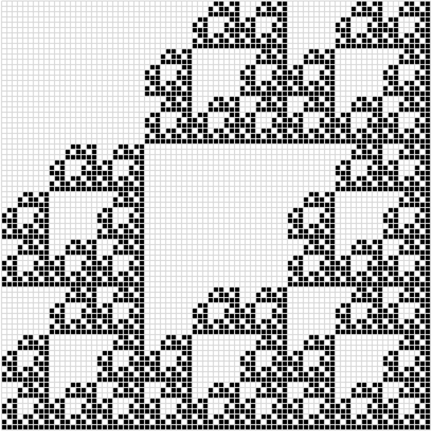
Conclusion

Weak grid



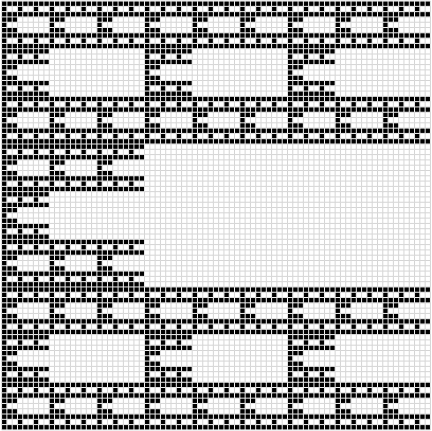
Undecidable

Weak grid



The proof is much harder than in the strong grid case.

Isthmus



We don't know anything about this one.

Conclusion

And about the Hausdorff dimension ?...

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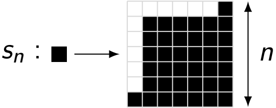

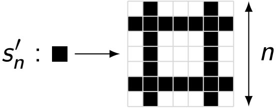

There is no threshold.

Thank you for your attention !