The group of reversible Turing machines

Sebastián Barbieri, Jarkko Kari and Ville Salo

LIP, ENS de Lyon – CNRS – INRIA – UCBL – Université de Lyon

University of Turku

Center for Mathematical Modeling, University of Chile

AUTOMATA
June, 2016

Motivation

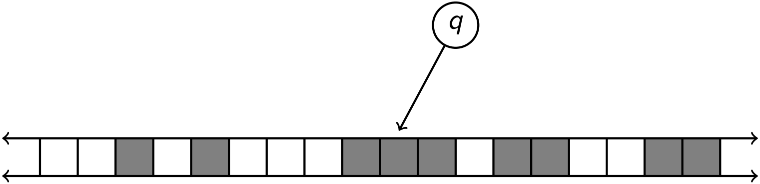
Recall that a Turing machine is defined by a rule :

$$\delta_T : \Sigma \times Q \to \Sigma \times Q \times \{-1, 0, 1\}$$

Motivation

Recall that a Turing machine is defined by a rule :

$$\delta_T : \Sigma \times Q \to \Sigma \times Q \times \{-1, 0, 1\}$$

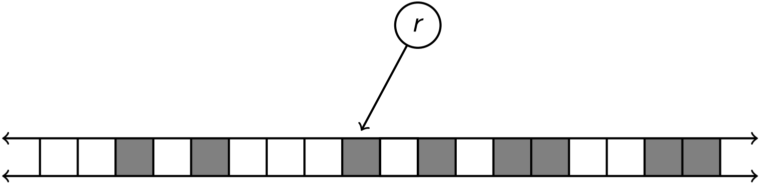


$$\delta_T(\blacksquare, q) = (\square, r, -1)$$

Motivation

Recall that a Turing machine is defined by a rule :

$$\delta_T : \Sigma \times Q \to \Sigma \times Q \times \{-1, 0, 1\}$$

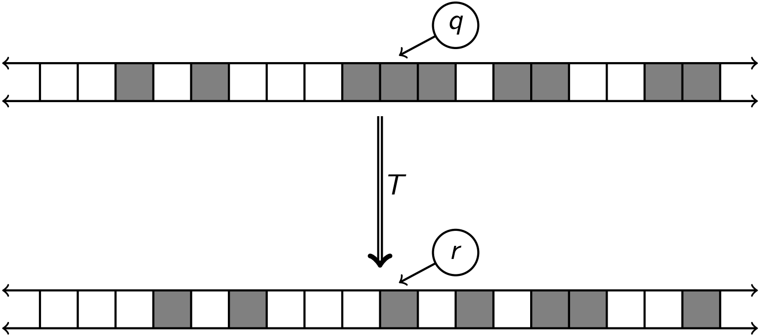


$$\delta_T(\blacksquare, q) = (\square, r, -1)$$

Motivation

This defines a natural action

$$T : \Sigma^{\mathbb{Z}} \times Q \to \Sigma^{\mathbb{Z}} \times Q$$



Motivation

This defines a natural action

$$T : \Sigma^{\mathbb{Z}} \times Q \to \Sigma^{\mathbb{Z}} \times Q$$

Such that if $(x, q) \in \Sigma^{\mathbb{Z}} \times Q$ and $\delta_T(x_0, q) = (a, r, d)$ then :

$$T(x, q) = (\sigma_{-d}(\tilde{x}), q')$$

where $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is the shift action given by $\sigma_d(x)_z = x_{z-d}$, $\tilde{x}_0 = a$ and $\tilde{x}|_{\mathbb{Z}\setminus\{0\}} = x|_{\mathbb{Z}\setminus\{0\}}$.

Motivation

- The composition of two actions $T \circ T'$ is not necessarily an action generated by a Turing machine.
- if the action T is a bijection then the inverse it not necessarily an action generated by a Turing machine.

Motivation

- The composition of two actions $T \circ T'$ is not necessarily an action generated by a Turing machine.
- if the action T is a bijection then the inverse it not necessarily an action generated by a Turing machine.

As in cellular automata, the class of CA with radius bounded by some $k \in \mathbb{N}$ is not closed under composition or inverses.

Definition

Let's get rid of these constrains. Given F, F' finite subsets of $\mathbb{Z}^d$, consider instead of δ_T a function :

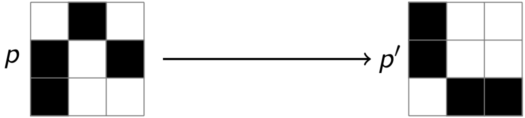
$$f_T : \Sigma^F \times Q \to \Sigma^{F'} \times Q \times \mathbb{Z}^d,$$

Definition

Let's get rid of these constrains. Given F, F' finite subsets of $\mathbb{Z}^d$, consider instead of δ_T a function :

$$f_T : \Sigma^F \times Q \to \Sigma^{F'} \times Q \times \mathbb{Z}^d,$$

Let $F = F' = \{0,1,2\}^2$, then $f_T(p,q) = (p', q', \vec{d})$ means :

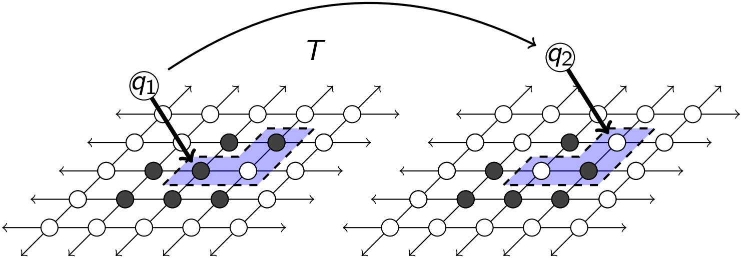
- 

- Turn state q into state q'
- Move head by $\vec{d}$.

Moving head model

f_T defines naturally an action

$$T \curvearrowright \Sigma^{\mathbb{Z}^d} \times Q \times \mathbb{Z}^d$$

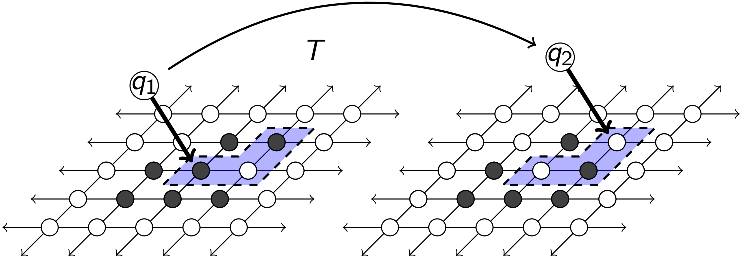


$f(\bullet\,\overset{\bullet}{\circ}, q_1) = (\circ\,\overset{\circ}{\bullet}, q_2, (1,1))$ $\quad F = \{(0,0),(1,0),(1,1)\}$

Moving head model

f_T defines naturally an action

$$T \curvearrowright \Sigma^{\mathbb{Z}^d} \times Q \times \mathbb{Z}^d$$



$f(\bullet\overset{\bullet}{\circ}, q_1) = (\circ\overset{\circ}{\bullet}, q_2, (1,1))$ $\quad F = \{(0,0),(1,0),(1,1)\}$

Let $|\Sigma| = n$ and $|Q| = k$.
$(\mathrm{TM}(\mathbb{Z}^d, n, k), \circ)$ is the monoid of all such T with the composition operation ; $(\mathrm{RTM}(\mathbb{Z}^d, n, k), \circ)$ is the group of all such T which are bijective .

Moving head model : As cellular automata

Let $Q = \{1, \ldots, k\}$ and $\Sigma = \{0, \ldots, n-1\}$.

$$\Sigma^{\mathbb{Z}^d} = \{x : \mathbb{Z}^d \to \Sigma\}$$

$$X_k = \{x \in \{0, 1, \ldots, k\}^{\mathbb{Z}^d} \mid 0 \notin \{x_{\vec{u}}, x_{\vec{v}}\} \implies \vec{u} = \vec{v}\}$$

Let $X_{n,k} = \Sigma^{\mathbb{Z}^d} \times X_k$ and $Y = \Sigma^{\mathbb{Z}^d} \times \{0^{\mathbb{Z}^d}\}$. Then :

Moving head model : As cellular automata

Let $Q = \{1, \ldots, k\}$ and $\Sigma = \{0, \ldots, n-1\}$.

$$\Sigma^{\mathbb{Z}^d} = \{x : \mathbb{Z}^d \to \Sigma\}$$

$$X_k = \{x \in \{0, 1, \ldots, k\}^{\mathbb{Z}^d} \mid 0 \notin \{x_{\vec{u}}, x_{\vec{v}}\} \implies \vec{u} = \vec{v}\}$$

Let $X_{n,k} = \Sigma^{\mathbb{Z}^d} \times X_k$ and $Y = \Sigma^{\mathbb{Z}^d} \times \{0^{\mathbb{Z}^d}\}$. Then :

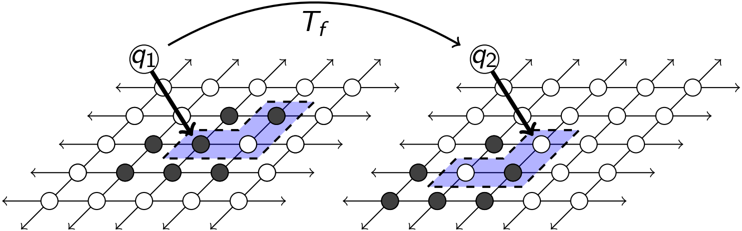
$$\mathrm{TM}(\mathbb{Z}^d, n, k) = \{\phi \in \mathrm{End}(X_{n,k}) \mid \phi|_Y = \mathrm{id}, \phi^{-1}(Y) = Y\}$$

$$\mathrm{RTM}(\mathbb{Z}^d, n, k) = \{\phi \in \mathrm{Aut}(X_{n,k}) \mid \phi|_Y = \mathrm{id}\}$$

Moving tape model

f_T defines naturally an action

$$T \curvearrowright \Sigma^{\mathbb{Z}^d} \times Q$$

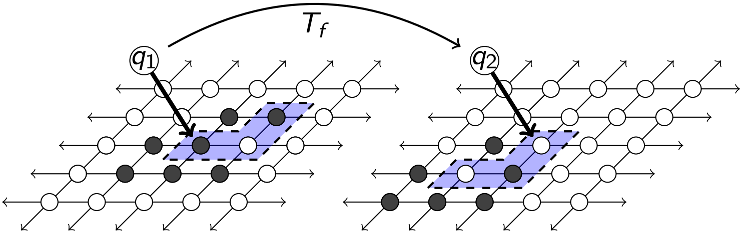


$f\left(\begin{smallmatrix} & \bullet \\ \bullet & \circ \end{smallmatrix}, q_1\right) = \left(\begin{smallmatrix} & \circ \\ \circ & \bullet \end{smallmatrix}, q_2, (1,1)\right) \qquad F = \{(0,0), (1,0), (1,1)\}$

Moving tape model

f_T defines naturally an action

$$T \curvearrowright \Sigma^{\mathbb{Z}^d} \times Q$$



$f(\bullet\overset{\bullet}{\circ}, q_1) = (\circ\overset{\circ}{\bullet}, q_2, (1,1))$ $\quad F = \{(0,0),(1,0),(1,1)\}$

Let $|\Sigma| = n$ and $|Q| = k$.
$(\mathrm{TM}_{\mathrm{fix}}(\mathbb{Z}^d, n, k), \circ)$ is the monoid of all such T with the composition operation ; $(\mathrm{RTM}_{\mathrm{fix}}(\mathbb{Z}^d, n, k), \circ)$ is the group of all such T which are bijective .

Moving tape model : dynamical definition

Let $x, y \in \Sigma^{\mathbb{Z}^d}$. x and y are *asymptotic*, and write $x \sim y$, if they differ in finitely many coordinates. We write $x \sim_m y$ if $x_{\vec{v}} = y_{\vec{v}}$ for all $|\vec{v}| \geq m$.

Moving tape model : dynamical definition

Let $x, y \in \Sigma^{\mathbb{Z}^d}$. x and y are *asymptotic*, and write $x \sim y$, if they differ in finitely many coordinates. We write $x \sim_m y$ if $x_{\vec{v}} = y_{\vec{v}}$ for all $|\vec{v}| \geq m$. Let $T : \Sigma^{\mathbb{Z}^d} \times Q \to \Sigma^{\mathbb{Z}^d} \times Q$ be a function.

T is a moving tape Turing machine $\iff$ T is continuous, and for a continuous function $s : \Sigma^{\mathbb{Z}^d} \times Q \to \mathbb{Z}^d$ and $a \in \mathbb{N}$ we have $T(x, q)_1 \sim_a \sigma_{s(x,q)}(x)$ for all $(x, q) \in \Sigma^{\mathbb{Z}^d} \times Q$.

Moving tape model : dynamical definition

Let $x, y \in \Sigma^{\mathbb{Z}^d}$. x and y are *asymptotic*, and write $x \sim y$, if they differ in finitely many coordinates. We write $x \sim_m y$ if $x_{\vec{v}} = y_{\vec{v}}$ for all $|\vec{v}| \geq m$. Let $T : \Sigma^{\mathbb{Z}^d} \times Q \to \Sigma^{\mathbb{Z}^d} \times Q$ be a function.

T is a moving tape Turing machine $\iff$ T is continuous, and for a continuous function $s : \Sigma^{\mathbb{Z}^d} \times Q \to \mathbb{Z}^d$ and $a \in \mathbb{N}$ we have $T(x, q)_1 \sim_a \sigma_{s(x,q)}(x)$ for all $(x, q) \in \Sigma^{\mathbb{Z}^d} \times Q$.

$s : \Sigma^{\mathbb{Z}^d} \times Q \to \mathbb{Z}^d$ is the shift indicator function

Equivalence of the models

$\mathrm{RTM}_{\mathrm{fix}}(\mathbb{Z}^d, 1, k) \cong S_k$ and $\mathbb{Z}^d \hookrightarrow \mathrm{RTM}(\mathbb{Z}^d, 1, k)$.

Equivalence of the models

$\mathrm{RTM}_{\mathrm{fix}}(\mathbb{Z}^d, 1, k) \cong S_k$ and $\mathbb{Z}^d \hookrightarrow \mathrm{RTM}(\mathbb{Z}^d, 1, k)$.

Proposition

If $n \geq 2$ then :

$$\mathrm{TM}_{\mathrm{fix}}(\mathbb{Z}^d, n, k) \cong \mathrm{TM}(\mathbb{Z}^d, n, k)$$
$$\mathrm{RTM}_{\mathrm{fix}}(\mathbb{Z}^d, n, k) \cong \mathrm{RTM}(\mathbb{Z}^d, n, k).$$

Properties of RTM

Proposition

Let $T \in \mathrm{TM}_{\mathrm{fix}}(\mathbb{Z}^d, n, k)$. Then the following are equivalent :

1. T is injective.
2. T is surjective.
3. $T \in \mathrm{RTM}_{\mathrm{fix}}(\mathbb{Z}^d, n, k)$.
4. T preserves the uniform measure ($\mu(T^{-1}(A)) = \mu(A)$ for all Borel sets A).
5. $\mu(T(A)) = \mu(A)$ for all Borel sets A.

Properties of RTM

Proposition

If $n \geq 2$ $\mathrm{RTM}(\mathbb{Z}^d, n, k)$ is not finitely generated.

Properties of RTM

Proposition

If $n \geq 2$ $\mathrm{RTM}(\mathbb{Z}^d, n, k)$ is not finitely generated.

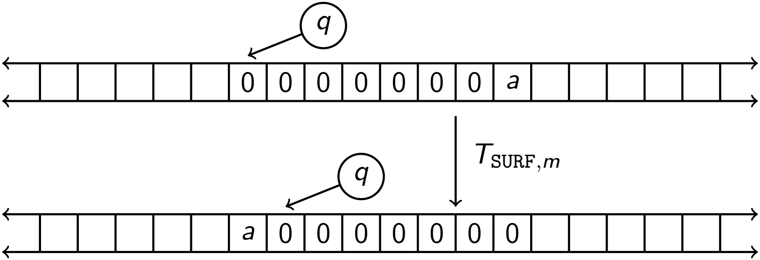
Proof : We find an epimorphism from RTM to a non-finitely generated group.
Let $T \in \mathrm{RTM}_{\mathrm{fix}}(\mathbb{Z}^d, n, k)$, therefore, it has a shift indicator $s : \Sigma^{\mathbb{Z}^d} \times Q \to \mathbb{Z}^d$. Define

$$\alpha(T) := \mathrm{E}_\mu(s) = \int_{\Sigma^{\mathbb{Z}^d} \times Q} s(x, q) d\mu,$$

One can check that $\alpha(T_1 \circ T_2) = \alpha(T_1) + \alpha(T_2)$.
Therefore $\alpha : \mathrm{RTM}(\mathbb{Z}^d, n, k) \to \mathbb{Q}^d$ is an homomorphism

Properties of RTM

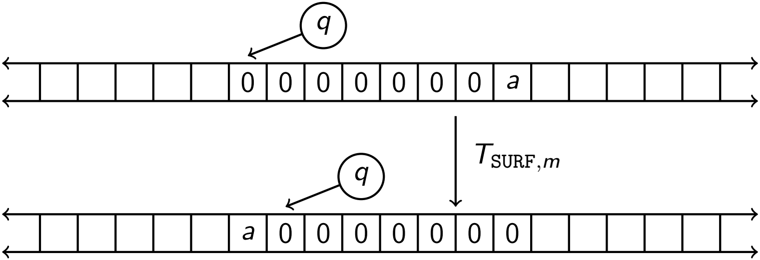
Now consider the machine $T_{\mathrm{SURF},m}$ where for all $a \in \Sigma$ and $q \in Q$:



$f(0^m a, q) = (a0^m, q, 1)$. Otherwise $f(u, q) = (u, q, 0)$.

Properties of RTM

Now consider the machine $T_{\mathtt{SURF},m}$ where for all $a \in \Sigma$ and $q \in Q$:

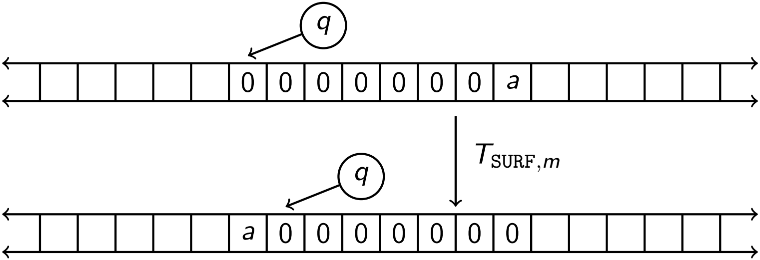


$$f(0^m a, q) = (a0^m, q, 1). \text{ Otherwise } f(u,q) = (u,q,0).$$

$T_{\mathtt{SURF},m} \in \mathrm{RTM}(\mathbb{Z}, n, k)$ and $\alpha(T_{\mathtt{SURF},m}) = 1/n^m$

Properties of RTM

Now consider the machine $T_{\mathrm{SURF},m}$ where for all $a \in \Sigma$ and $q \in Q$:



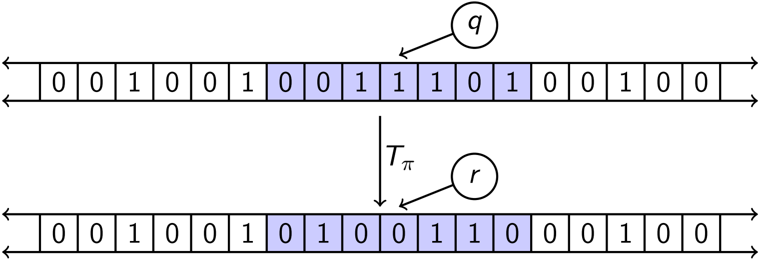
$$f(0^m a, q) = (a0^m, q, 1). \text{ Otherwise } f(u, q) = (u, q, 0).$$

$T_{\mathrm{SURF},m} \in \mathrm{RTM}(\mathbb{Z}, n, k)$ and $\alpha(T_{\mathrm{SURF},m}) = 1/n^m$

$\langle (1/n^m)_{m \in \mathbb{N}} \rangle \subset \alpha(\mathrm{RTM}(\mathbb{Z}, n, k))$ which is thus a non-finitely generated subgroup of $\mathbb{Q}$.

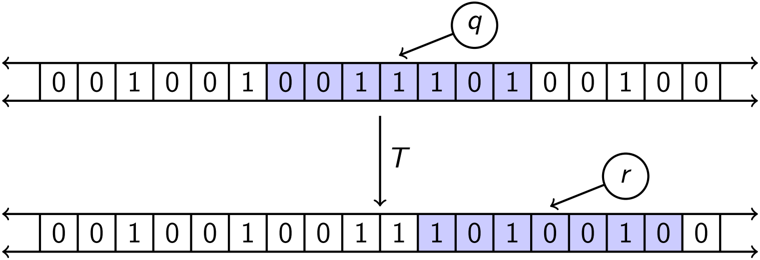
Interesting subgroups of RTM

▷ $\mathrm{LP}(\mathbb{Z}^d, n, k) \longrightarrow$ Local permutations.



Interesting subgroups of RTM

- ▷ $\mathrm{LP}(\mathbb{Z}^d, n, k) \longrightarrow$ Local permutations.
- ▷ $\mathrm{RFA}(\mathbb{Z}^d, n, k) \longrightarrow$ Reversible finite-state automata.



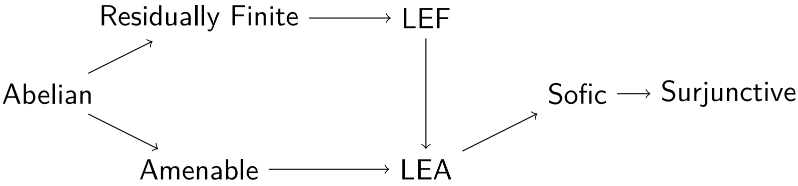
Interesting subgroups of RTM

- $\triangleright$ $\mathrm{LP}(\mathbb{Z}^d, n, k) \longrightarrow$ Local permutations.
- $\triangleright$ $\mathrm{RFA}(\mathbb{Z}^d, n, k) \longrightarrow$ Reversible finite-state automata.
- $\triangleright$ $\mathrm{OB}(\mathbb{Z}^d, n, k) \longrightarrow$ Oblivous machines $\langle \mathrm{LP}, \mathrm{Shift} \rangle$.

Interesting subgroups of RTM

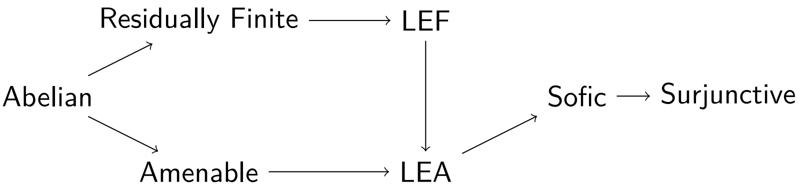
- $\triangleright$ $\mathrm{LP}(\mathbb{Z}^d, n, k) \longrightarrow$ Local permutations.
- $\triangleright$ $\mathrm{RFA}(\mathbb{Z}^d, n, k) \longrightarrow$ Reversible finite-state automata.
- $\triangleright$ $\mathrm{OB}(\mathbb{Z}^d, n, k) \longrightarrow$ Oblivous machines $\langle \mathrm{LP}, \mathrm{Shift} \rangle$.
- $\triangleright$ $\mathsf{EL}(\mathbb{Z}^d, n, k) \longrightarrow$ Elementary machines $\langle \mathrm{LP}, \mathrm{RFA} \rangle$.

Small group theory roadmap



- Res. finite groups are those where every non-identity element can be mapped to a non-identity element by a homomorphism to a finite group
- Amenable groups admit left invariant finitely additive measures.
- LEF and LEA stand for locally embeddable into (finite/amenable) groups.
- Sofic groups are generalizations of LEF and LEA.
- Surjunctive groups satisfy that all injective CA are surjective.

Small group theory roadmap



Theorem

$\forall n \geq 2, \mathrm{RTM}(\mathbb{Z}^d, n, k)$ is LEF but neither amenable nor residually finite.

Some properties : $\mathrm{LP}(\mathbb{Z}^d, n, k)$

For $n \geq 2$, we have $S_\infty \hookrightarrow \mathrm{LP}(\mathbb{Z}^d, n, k)$.

Some properties : $\mathrm{LP}(\mathbb{Z}^d, n, k)$

For $n \geq 2$, we have $S_\infty \hookrightarrow \mathrm{LP}(\mathbb{Z}^d, n, k)$.

This means that RTM is not residually finite, and that it contains all finite groups.

Some properties : $\mathrm{LP}(\mathbb{Z}^d, n, k)$

For $n \geq 2$, we have $S_\infty \hookrightarrow \mathrm{LP}(\mathbb{Z}^d, n, k)$.

This means that RTM is not residually finite, and that it contains all finite groups.

$\mathrm{LP}(\mathbb{Z}^d, n, k)$ is locally finite and amenable.

Some properties : $\mathrm{LP}(\mathbb{Z}^d, n, k)$

For $n \geq 2$, we have $S_\infty \hookrightarrow \mathrm{LP}(\mathbb{Z}^d, n, k)$.

This means that RTM is not residually finite, and that it contains all finite groups.

$\mathrm{LP}(\mathbb{Z}^d, n, k)$ is locally finite and amenable.

In particular, for $n \geq 2$ $\mathrm{LP}(\mathbb{Z}^d, n, k)$ is not finitely generated.

Some properties : $\mathrm{OB}(\mathbb{Z}^d, n, k)$

Now let's add the shift. Recall that $\mathrm{OB}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}, \mathrm{Shift} \rangle$.

Some properties : $\mathrm{OB}(\mathbb{Z}^d, n, k)$

Now let's add the shift. Recall that $\mathrm{OB}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}, \mathrm{Shift} \rangle$.

$\mathrm{OB}(\mathbb{Z}^d, n, k)$ is amenable.

Some properties : $\mathrm{OB}(\mathbb{Z}^d, n, k)$

Now let's add the shift. Recall that $\mathrm{OB}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}, \mathrm{Shift} \rangle$.

$\mathrm{OB}(\mathbb{Z}^d, n, k)$ is amenable.

Proof : α gives a short exact sequence

$$1 \longrightarrow \mathrm{LP}(\mathbb{Z}^d, n, k) \longrightarrow \mathrm{OB}(\mathbb{Z}^d, n, k) \longrightarrow \mathbb{Z}^d \longrightarrow 1.$$

Some properties : $\mathrm{OB}(\mathbb{Z}^d, n, k)$

Now let's add the shift. Recall that $\mathrm{OB}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}, \mathrm{Shift} \rangle$.

$\mathrm{OB}(\mathbb{Z}^d, n, k)$ is amenable.

Proof : α gives a short exact sequence

$$1 \longrightarrow \mathrm{LP}(\mathbb{Z}^d, n, k) \longrightarrow \mathrm{OB}(\mathbb{Z}^d, n, k) \longrightarrow \mathbb{Z}^d \longrightarrow 1.$$

$\mathrm{OB}(\mathbb{Z}^d, n, k)$ contains all generalized lamplighter groups $G \wr \mathbb{Z}^d$.

Some properties : $\mathrm{OB}(\mathbb{Z}^d, n, k)$

Now let's add the shift. Recall that $\mathrm{OB}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}, \mathrm{Shift} \rangle$.

$\mathrm{OB}(\mathbb{Z}^d, n, k)$ is amenable.

Proof : α gives a short exact sequence

$$1 \longrightarrow \mathrm{LP}(\mathbb{Z}^d, n, k) \longrightarrow \mathrm{OB}(\mathbb{Z}^d, n, k) \longrightarrow \mathbb{Z}^d \longrightarrow 1.$$

$\mathrm{OB}(\mathbb{Z}^d, n, k)$ contains all generalized lamplighter groups $G \wr \mathbb{Z}^d$.

Theorem

$\mathrm{OB}(\mathbb{Z}^d, n, k)$ is finitely generated.

$\mathrm{OB}(\mathbb{Z}^d, n, k)$ is finitely generated.

This proof is based on the existence of strongly universal reversible gates for permutations of Σ^m.
A controlled swap is a transposition (s,t) where s,t have Hamming distance 1 in $Q \times \Sigma^m$.

Theorem

The group generated by the applications of controlled swaps of $Q \times \Sigma^4$ at arbitrary positions generates $Sym(Q \times \Sigma^m)$ if $|\Sigma|$ is odd and $Alt(Q \times \Sigma^m)$ if it's even.
Corollary : $[Sym(Q \times \Sigma^m)]_{m+1} \subset \langle [Sym(Q \times \Sigma^4)]_{m+1} \rangle$.

$\mathrm{OB}(\mathbb{Z}^d, n, k)$ is finitely generated.

Using this result, a generating set can be constructed :

- $A_1 =$ Shifts T_{e_i} for $\{e_i\}_{i \leq d}$ a base of $\mathbb{Z}^d$.
- $A_2 =$All $T_\pi \in \mathrm{LP}(\mathbb{Z}^d, n, k)$ of fixed support $E \subset \mathbb{Z}^d$ of size 4.
- $A_3 =$ The swaps of symbols in positions $(\vec{0}, e_i)$.

Some properties : $\mathrm{RFA}(\mathbb{Z}^d, n, k)$

Recall that $\mathrm{RFA}(\mathbb{Z}^d, n, k)$ is the subgroup of machines which do not modify the tape.

For $n \geq 2$, $\mathrm{RFA}(\mathbb{Z}^d, n, k)$ contains all countable free groups.

Some properties : $\mathrm{RFA}(\mathbb{Z}^d, n, k)$

Recall that $\mathrm{RFA}(\mathbb{Z}^d, n, k)$ is the subgroup of machines which do not modify the tape.

For $n \geq 2$, $\mathrm{RFA}(\mathbb{Z}^d, n, k)$ contains all countable free groups.

In particular, this means that RFA and RTM are not amenable.

Some properties : $\mathrm{RFA}(\mathbb{Z}^d, n, k)$

Recall that $\mathrm{RFA}(\mathbb{Z}^d, n, k)$ is the subgroup of machines which do not modify the tape.

For $n \geq 2$, $\mathrm{RFA}(\mathbb{Z}^d, n, k)$ contains all countable free groups.

In particular, this means that RFA and RTM are not amenable.

$\mathrm{RFA}(\mathbb{Z}^d, n, k)$ is residually finite but not finitely generated.

Some properties : $\mathrm{EL}(\mathbb{Z}^d, n, k)$ and $\mathrm{RTM}(\mathbb{Z}^d, n, k)$

Recall that $\mathrm{EL}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}(\mathbb{Z}^d, n, k), \mathrm{RFA}(\mathbb{Z}^d, n, k) \rangle$ is the subgroup of elementary Turing machines.
Question : Is $\mathrm{EL}(\mathbb{Z}^d, n, k) = \mathrm{RTM}(\mathbb{Z}^d, n, k)$?

Some properties : $\mathrm{EL}(\mathbb{Z}^d, n, k)$ and $\mathrm{RTM}(\mathbb{Z}^d, n, k)$

Recall that $\mathrm{EL}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}(\mathbb{Z}^d, n, k), \mathrm{RFA}(\mathbb{Z}^d, n, k) \rangle$ is the subgroup of elementary Turing machines.
Question : Is $\mathrm{EL}(\mathbb{Z}^d, n, k) = \mathrm{RTM}(\mathbb{Z}^d, n, k)$?

For $n \geq 2$, $\alpha(\mathrm{EL}(\mathbb{Z}^d, n, k)) = \alpha(\mathrm{RFA}(\mathbb{Z}^d, n, k))$ has bounded denominator.

Some properties : $\mathsf{EL}(\mathbb{Z}^d, n, k)$ and $\mathrm{RTM}(\mathbb{Z}^d, n, k)$

Recall that $\mathsf{EL}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}(\mathbb{Z}^d, n, k), \mathrm{RFA}(\mathbb{Z}^d, n, k) \rangle$ is the subgroup of elementary Turing machines.
Question : Is $\mathsf{EL}(\mathbb{Z}^d, n, k) = \mathrm{RTM}(\mathbb{Z}^d, n, k)$?

For $n \geq 2$, $\alpha(\mathsf{EL}(\mathbb{Z}^d, n, k)) = \alpha(\mathrm{RFA}(\mathbb{Z}^d, n, k))$ has bounded denominator.

In particular, this means that $\mathsf{EL} \subsetneq \mathrm{RTM}$.

Some properties : $\mathsf{EL}(\mathbb{Z}^d, n, k)$ and $\mathrm{RTM}(\mathbb{Z}^d, n, k)$

Recall that $\mathsf{EL}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}(\mathbb{Z}^d, n, k), \mathrm{RFA}(\mathbb{Z}^d, n, k) \rangle$ is the subgroup of elementary Turing machines.
Question : Is $\mathsf{EL}(\mathbb{Z}^d, n, k) = \mathrm{RTM}(\mathbb{Z}^d, n, k)$?

For $n \geq 2$, $\alpha(\mathsf{EL}(\mathbb{Z}^d, n, k)) = \alpha(\mathrm{RFA}(\mathbb{Z}^d, n, k))$ has bounded denominator.

In particular, this means that $\mathsf{EL} \subsetneq \mathrm{RTM}$.
Open : Is $\mathsf{EL} = \langle \mathsf{Ker}_\alpha(\mathrm{RTM}), \mathrm{Shift} \rangle$?

Some properties : $\mathrm{EL}(\mathbb{Z}^d, n, k)$ and $\mathrm{RTM}(\mathbb{Z}^d, n, k)$

Recall that $\mathrm{EL}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}(\mathbb{Z}^d, n, k), \mathrm{RFA}(\mathbb{Z}^d, n, k) \rangle$ is the subgroup of elementary Turing machines.
Question : Is $\mathrm{EL}(\mathbb{Z}^d, n, k) = \mathrm{RTM}(\mathbb{Z}^d, n, k)$?

For $n \geq 2$, $\alpha(\mathrm{EL}(\mathbb{Z}^d, n, k)) = \alpha(\mathrm{RFA}(\mathbb{Z}^d, n, k))$ has bounded denominator.

In particular, this means that $\mathrm{EL} \subsetneq \mathrm{RTM}$.
Open : Is $\mathrm{EL} = \langle \mathrm{Ker}_\alpha(\mathrm{RTM}), \mathrm{Shift} \rangle$?

RTM is a LEF group, in particular, it is sofic.

Computability properties

Given a finite rules : f, f' :

- It is decidable (in any model) whether $T_f = T_{f'}$.
- We can effectively calculate a rule for $T_f \circ T_{f'}$.
- It is decidable whether T_f is reversible.
- If it is, we can effectively compute a rule for T_f^{-1}.

Computability properties

Given a finite rules : f, f' :

- It is decidable (in any model) whether $T_f = T_{f'}$.
- We can effectively calculate a rule for $T_f \circ T_{f'}$.
- It is decidable whether T_f is reversible.
- If it is, we can effectively compute a rule for T_f^{-1}.

$\mathrm{RTM}(\mathbb{Z}^d, n, k)$ is a recursively presented group with decidable word problem.

Computability properties

Given a finite rules : f, f' :

- It is decidable (in any model) whether $T_f = T_{f'}$.
- We can effectively calculate a rule for $T_f \circ T_{f'}$.
- It is decidable whether T_f is reversible.
- If it is, we can effectively compute a rule for T_f^{-1}.

$\mathrm{RTM}(\mathbb{Z}^d, n, k)$ is a recursively presented group with decidable word problem.

What can we say about the torsion ($\exists n$ such that $T^n = 1$) problem ?

Computability properties

Given a finite rules : f, f' :

- It is decidable (in any model) whether $T_f = T_{f'}$.
- We can effectively calculate a rule for $T_f \circ T_{f'}$.
- It is decidable whether T_f is reversible.
- If it is, we can effectively compute a rule for T_f^{-1}.

$\mathrm{RTM}(\mathbb{Z}^d, n, k)$ is a recursively presented group with decidable word problem.

What can we say about the torsion ($\exists n$ such that $T^n = 1$) problem ?
It is undecidable in $\mathrm{RTM}(\mathbb{Z}^d, n, k)$ if $n \geq 2$. What about RFA ?

The torsion problem for RFA

$\mathrm{RFA}(\mathbb{Z}, n, k)$ has decidable torsion problem.

Proof : As $\mathbb{Z}$ is two-ended, any non-torsion machine must shift to the left or right in at least a periodic configuration.

The torsion problem for RFA

$\mathrm{RFA}(\mathbb{Z}, n, k)$ has decidable torsion problem.

Proof : As $\mathbb{Z}$ is two-ended, any non-torsion machine must shift to the left or right in at least a periodic configuration.

Theorem

$\mathrm{RFA}(\mathbb{Z}^d, n, k)$ has undecidable torsion problem for $d, n \geq 2$.

The torsion problem for RFA

$\mathrm{RFA}(\mathbb{Z}, n, k)$ has decidable torsion problem.

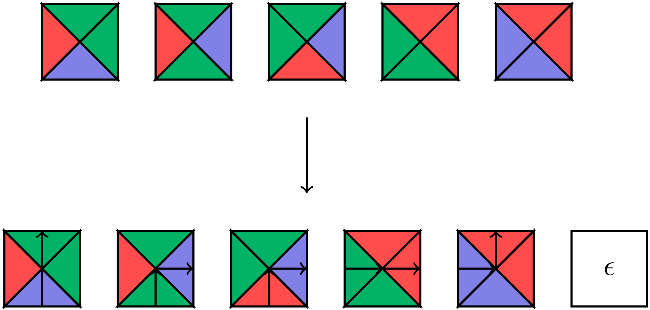
Proof : As $\mathbb{Z}$ is two-ended, any non-torsion machine must shift to the left or right in at least a periodic configuration.

Theorem

$\mathrm{RFA}(\mathbb{Z}^d, n, k)$ has undecidable torsion problem for $d, n \geq 2$.

Proof : Reduction to the snake tiling problem, which reduces to the domino problem for $\mathbb{Z}^d$.

The snake problem



Can we tile the plane in a way which produces a bi-infinite path ?

The snake problem

Theorem (Kari)

The snake tiling problem is undecidable.

The proof uses a plane filling curve generated by a substitution.

The snake problem

Theorem (Kari)

The snake tiling problem is undecidable.

The proof uses a plane filling curve generated by a substitution.

For every instance of the snake tiling problem, one can construct $T \in \mathrm{RFA}$ which walks the path of the snake, and turns back if it encounters a problem.

Thank you for your attention !