Effectiveness in finitely generated groups

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Outline

$\mathbb{Z}$-Subshifts

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- $\mathcal{A}^{\mathbb{Z}}$ is the set of bi-infinite words on $\mathcal{A}$.

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$$X = X_{\mathcal{F}} := \left\{ x \in \mathcal{A}^{\mathbb{Z}} \mid \forall n \in \mathbb{Z}, k \in \mathbb{N}_0, x_n \dots x_{n+k} \notin \mathcal{F} \right\}.$$

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Example : Fibonacci shift. Let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \{11\}$. Then $X_{\mathcal{F}}$ is the set of all bi-infinite words which have no pairs of consecutive 1's.

$$x = \dots 010100010100100100 \dots \in X_{\mathcal{F}}$$

$\mathbb{Z}$-Subshifts

Example : one-or-less subshift

$$X_{\leq 1} := \{x \in \{0,1\}^{\mathbb{Z}} \mid |\{n \in \mathbb{Z} : x_n = 1\}| \leq 1\}.$$

Is a $\mathbb{Z}$-subshift as it is defined by the set $\mathcal{F} = \{10^n1 | n \in \mathbb{N}_0\}$.

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Generalize the notion to $\mathbb{Z}^d$

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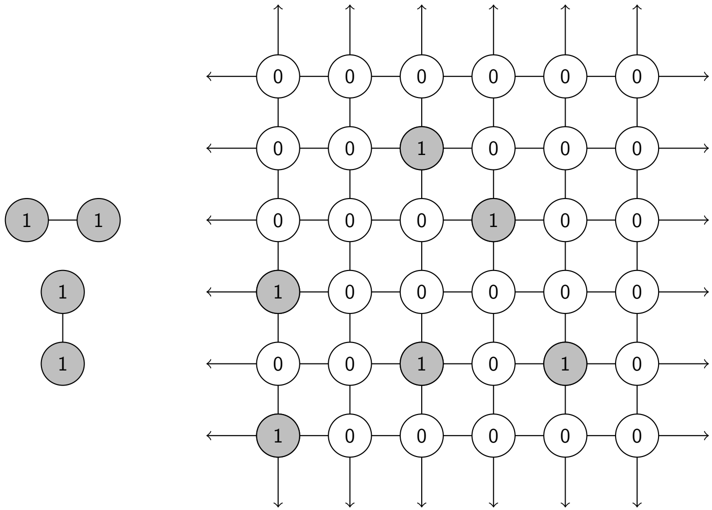

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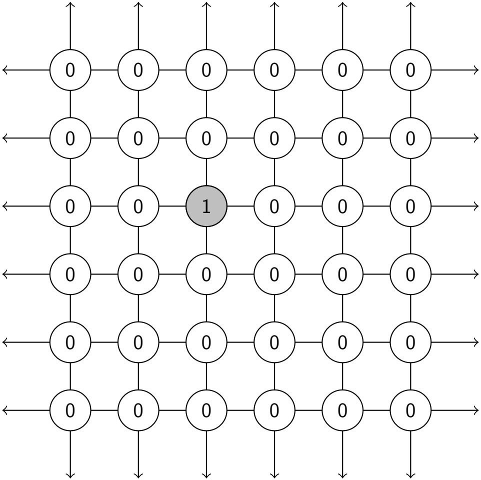

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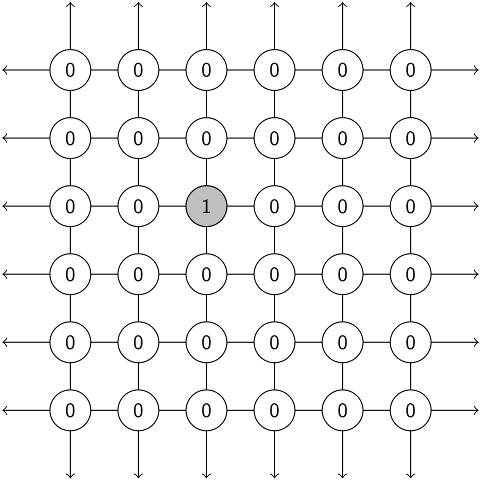

Question : What if we want to go further ? What is a good base structure ?

G-subshifts

Definition

Let G be a group. A G-subshift is a set $X \subset \mathcal{A}^G$ such that there exists a set of forbidden patterns $\mathcal{F} \subset \mathcal{A}^*_G$ where $\mathcal{A}^*_G := \bigcup_{F \subset G, |F| < \infty} \mathcal{A}^F$ such that :

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Where the shift action $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is such that

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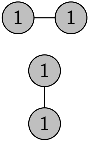
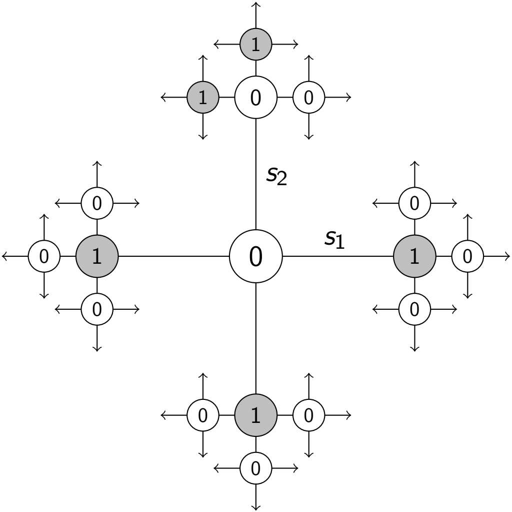
Example : S-Fibonacci shift. Let $\mathcal{A} = \{0,1\}$, $S \subset G$ a finite generator of G and $\mathcal{F} = \{1^{\{1_G, s\}}, s \in S\}$ then $X_{fib,S} = X_{\mathcal{F}}$ is the S-Fibonacci shift.

Example : S-Fibonacci shift for $G = F_2$

$\mathcal{F} =$ 



Interesting classes

G-SFTs

A G-subshift X over $\mathcal{A}$ is said to be a G-subshift of finite type (G-SFT) if there exists a finite set of patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

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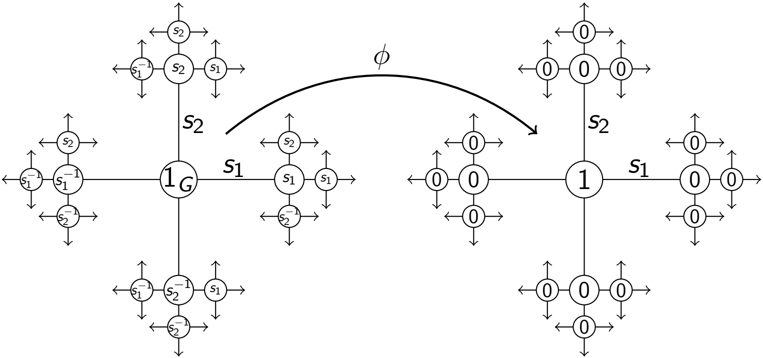
Sofic G-subshifts

A G-subshift Y over $\mathcal{A}$ is said to be a sofic G-subshift if there exists a G-SFT X and a local surjective sliding block code. That is : $\Phi : \mathcal{A}_X^F \to \mathcal{A}_Y$ such that $\phi : X \to Y$ defined by $\phi(x)_g = \Phi(\sigma_{g^{-1}}(x)|_F)$ is surjective.

Example : S-Fibonacci shift. For every group G generated by a finite set S the S-Fibonacci shift is a G-SFT.

$X_{\leq 1}$ is a sofic F_2-subshift.



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Example : Context-free subshift. Consider $\mathcal{A} = \{a, b, c\}$, $\mathcal{F} = \{ab^k c^l a | k, l \in \mathbb{N}_0, k \neq l\}$. The subshift $X = X_{\mathcal{F}}$ is the context free subshift. It is not a sofic $\mathbb{Z}$-subshift but it is effective.

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Definition :

A $\mathbb{Z}^d$-subshift $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is said to be effective if there is a set $\mathcal{F} \subset \mathcal{A}^*_{\mathbb{Z}^d}$ such that $X = X_{\mathcal{F}}$ and a Turing machine which accepts a coding if and only if it is both consistent and the pattern it codes belongs to $\mathcal{F}$.

Pattern codings

Question : How can one generalize such a coding for an arbitrary finitely generated group G ?

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Definition : Pattern Coding

Let $S \subset G$ be a finite generator. A pattern coding c is a finite set of tuples $c = (w_i, a_i)_{1 \leq i \leq n}$ where $w_i \in (S \cup S^{-1})^*$ and $a_i \in \mathcal{A}$.
c is consistent if for every pair of tuples w_i, w_j which represent the same element in G then $a_i = a_j$.
For a consistent pattern coding c we associate the pattern $\Pi(c) \in \mathcal{A}_G^*$ such that $supp(\Pi(c)) = \bigcup_{i \in I} w_i$ and $\Pi(c)_{w_i} = a_i$.

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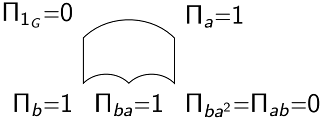

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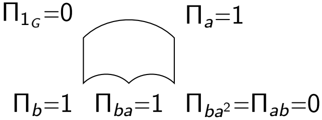

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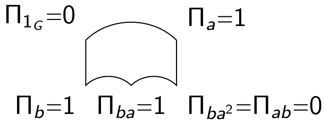

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  is *inconsistent* since $abab^{-1}$ and $bab^{-1}a$ represent the same element.

First approach : $\mathbb{Z}$-effectiveness

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A G-subshift $X \subset \mathcal{A}^G$ is *$\mathbb{Z}$-effective* if there is $\mathcal{F} \subset \mathcal{A}_G^*$ such that $X = X_{\mathcal{F}}$ and a Turing machine T that accepts a pattern coding c if and only if it is either inconsistent or $\Pi(c) \in \mathcal{F}$.

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Question : Is it always possible to recognize if a pattern coding is inconsistent ?

Limitations of $\mathbb{Z}$-effectiveness

Definition : Word problem

Let $S \subset G$ be a finite generator of G. The word problem of G asks whether two words on $S \cup S^{-1}$ are equivalent in G. Formally :

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Example : Decidable word problem. The word problem for $\mathbb{Z}^2 \simeq \langle a, b | ab = ba \rangle$ is :

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Example : Undecidable word problem. If $f : \mathbb{N} \to \{0, 1\}$ is non-computable the group $G = \langle a, b, c, d \mid ab^n = c^n d, n \in f^{-1}(1) \rangle$ has undecidable word problem.

Limitations of $\mathbb{Z}$-effectiveness

Finitely generated groups

A finitely generated group G is said to be :

- Finitely presented if there is a presentation $G \simeq \langle S, R \rangle$ where both S and R are finite.
- Recursively presented if there is a presentation $G \simeq \langle S, R \rangle$ where S is finite and R is recognizable.

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Remark [Theorem : Novikov(55), Boone(58)]

There are finitely presented groups with undecidable word problem !

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Theorem

Let $|\mathcal{A}| \geq 2$ then the following are equivalent :

- *G is recursively presented.*
- *The $WP(G)$ is recognizable.*
- *The set of inconsistent patterns codings is recognizable.*

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Theorem

The one-or-less subshift :

$$X_{\leq 1} := \{x \in \{0,1\}^G \mid |\{g \in G : x_g = 1\}| \leq 1\}.$$

is not $\mathbb{Z}$-effective if $WP(G)$ is undecidable.

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Definition : G-machine

A G-machine is a Turing machine whose tape has been replaced by the group G. The transition function is $\delta : Q \times \Sigma \to Q \times \Sigma \times (S \cup S^{-1} \cup \{1_G\})$ where S is a finite set of generators of G.

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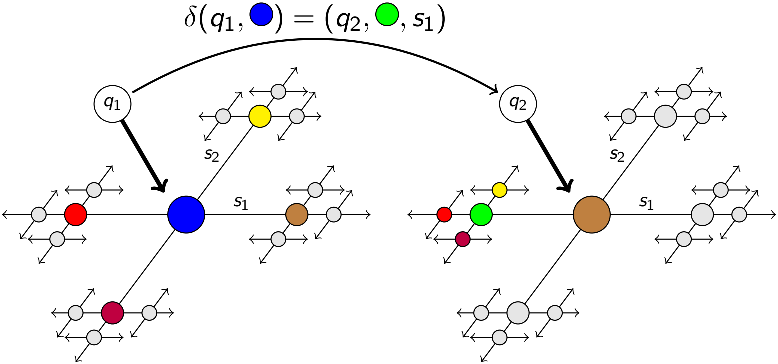
Remark : Computation is over patterns of Σ_G^* instead of Σ^*.

Example : Transition in a F_2-machine



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Definition :

- A set of patterns $\mathcal{P} \subseteq \mathcal{A}_G^*$ is said to be recognizable if there is a G-machine which accepts if and only if $P \in \mathcal{P}$.
- A set of patterns $\mathcal{P} \subseteq \mathcal{A}_G^*$ is said to be decidable if there is a G-machine which accepts if $P \in \mathcal{P}$ and rejects otherwise.

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G-effectiveness

A G-subshift $X \subset \mathcal{A}^G$ is G-effective if there exists a set of forbidden patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$ and $\mathcal{F}$ is G-recognizable.

Remark : The set of forbidden patterns $\mathcal{F}$ can be chosen to be maximal.

What can we say about G-effectiveness ?

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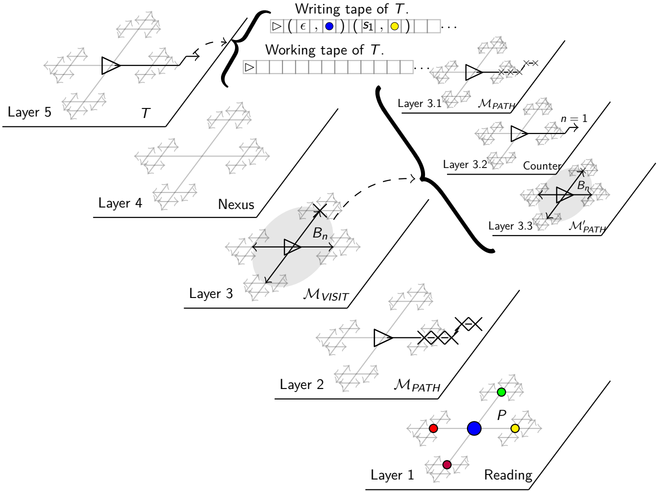
- Initiate a backtracking over G in order to mark a one sided-infinite path.
- Use the path to simulate one-sided Turing machines.

The construction for the previous theorem.



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- *Every G-SFT is G-effective.*
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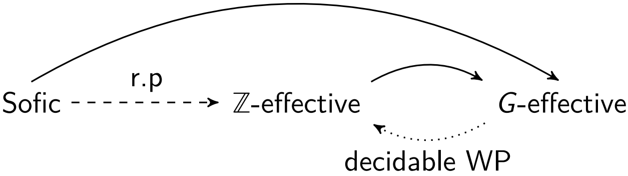

Outline

For which groups are there G-effective subshifts which are not sofic ?

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Question : Is it possible to construct G-effective subshifts which are not sofic in big classes of groups ?

Amenable groups

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- If G is finitely generated, the net can just be seen as a sequence.
- If G is generated by a finite set $S \subset G$, amenability reduces to :

$$\inf_{F \subset G, |F| < \infty} |\partial F| / |F| = 0.$$

Amenability

Examples of amenable groups

- Finite groups.
- Abelian groups ($\mathbb{Z}^d$).
- Nilpotent groups (Heisenberg group).
- Groups of sub-exponential growth (Grigorchuk group).
- Solvable groups ($BS(1,2)$, lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$).

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Examples of non-amenable groups

- Free groups.
- Groups containing F_2 as a subgroup.
- Tarksi monsters (counterexamples to Von Neumann's conjecture).

Amenability

Second case :

For every infinite, amenable and finitely generated group G there are G-effective subshifts which are not sofic.

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Proof : Similar to the one for the mirror shift.

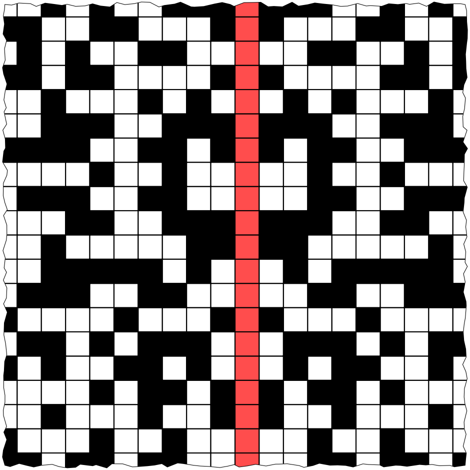
Mirror shift

Let $\mathcal{A} = \{\square, \blacksquare, {\color{red}\blacksquare}\}$ and consider the following set of forbidden patterns F_{mirror} :

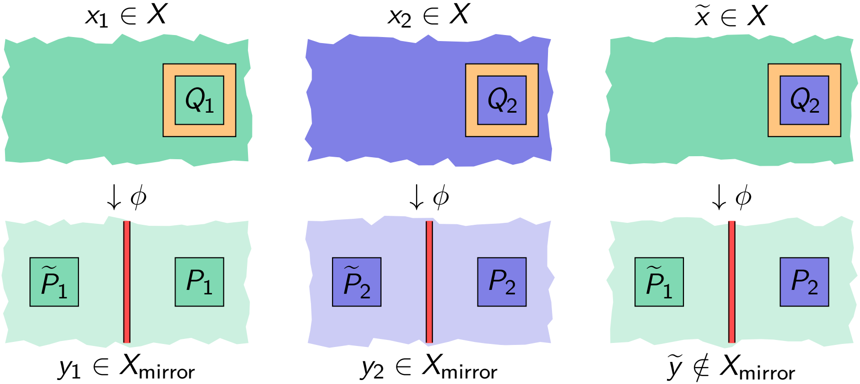
$$\left\{ \begin{smallmatrix}\square\\ {\color{red}\blacksquare}\end{smallmatrix}, \begin{smallmatrix}\blacksquare\\ {\color{red}\blacksquare}\end{smallmatrix}, \begin{smallmatrix}{\color{red}\blacksquare}\\ \square\end{smallmatrix}, \begin{smallmatrix}{\color{red}\blacksquare}\\ \blacksquare\end{smallmatrix} \right\} \cup \bigcup_{w \in \mathcal{A}^*} \{ {\color{red}\blacksquare}\, w\, {\color{red}\blacksquare}, \blacksquare\, w\, {\color{red}\blacksquare}\, \tilde{w}\, \square, \square\, w\, {\color{red}\blacksquare}\, \tilde{w}\, \blacksquare \}$$

where $\tilde{w}$ denotes the mirror image of the word w, which is the word of length $|w|$ defined by $(\tilde{w})_i = w_{|w|-i+1}$ for all $1 \leq i \leq |w|$.

Mirror shift



Proof that the mirror shift is not sofic



Amenable case : Ball mimic subshift

$\mathcal{G} = (g_i)_{i\in\mathbb{N}} \subset G$ and $\mathcal{H} = (h_i)_{i\in\mathbb{N}} \subset G$ be two sequences such that :

- The sets $(g_i B_i)_{i\in\mathbb{N}}$ and $(h_i B_i)_{i\in\mathbb{N}}$ are pairwise disjoint.
- They don't contain 1_G.

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Definition :

The ball mimic subshift $X_B(\mathcal{G}, \mathcal{H}) \subset \{0, 1, \boxtimes\}^G$ is G-subshift such that in every configuration $x \in X_B(\mathcal{G}, \mathcal{H})$ the symbol $\boxtimes$ appears at most once, and if for $\bar{g} \in G$ $x_{\bar{g}} = \boxtimes$ then $\forall i \in \mathbb{N}$:

$$\sigma_{(\bar{g}g_i)^{-1}}(x)|_{B_i} = \sigma_{(\bar{g}h_i)^{-1}}(x)|_{B_i}$$

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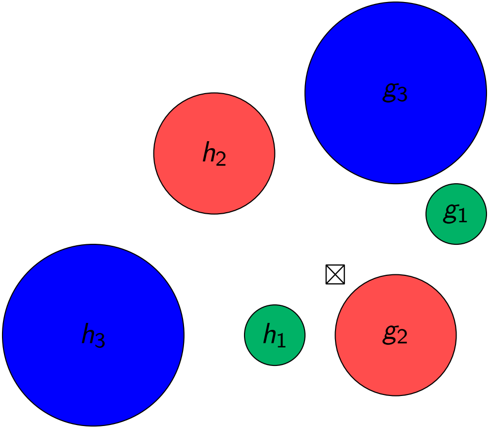

Groups with more than two ends

Ends in a group

Let G be a group generated by a finite set $S \subset G$. The number of ends $e(G)$ of the group G is the limit as n tends to infinity of the number of infinite connected components of $\Gamma(G, S) \setminus B_n$.

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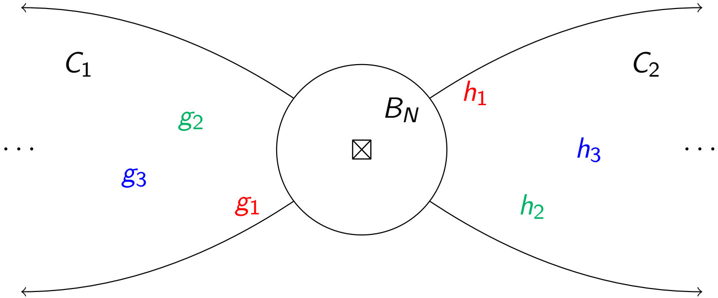
- $e(G) = 2$ if and only if G is infinite and virtually cyclic.
- If $e(G) = \infty$ Stallings theorem implies that G contains a non-abelian free group.
- Every virtually free group satisfies $e(G) \geq 2$.

Groups with more than two ends

Third case :

For every finitely generated group G such that $e(G) \geq 2$ there are G-effective subshifts which are not sofic.

The mimic subshift



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- A natural notion for effectiveness in finitely generated groups.

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Question : Is it true that for every infinite and finitely generated group G the class of G-effective subshifts is strictly larger than the class of sofic G-subshifts.

Merci beaucoup pour votre attention !

Avez-vous des questions ?