A remark on inverse limits of effective subshifts

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Abstract

We show that, for every finitely generated group with decidable word problem and undecidable domino problem, there exists a sequence of effective subshifts whose inverse limit is not the topological factor of any effective dynamical system. This follows from considerations on the universality under topological factors for this class of dynamical systems.

Keywords: symbolic dynamics, effective dynamical systems, universality, topological factors.

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1 Introduction

Given a finitely generated group G and a compact metrizable space X, one can consider the space of all actions of G on X by homeomorphisms. This space can be quite large, and a natural restriction is to consider only actions which are *effective*, in the sense that both the space and the dynamics can be described by an algorithm.

Effective actions can be formally defined using computable analysis. We say that an action of a finitely generated group $G \curvearrowright X$ is an *effective dynamical system* (EDS), if it is topologically conjugate to an action of G on a recursively compact subset of a computable metric space, in such a way that every generator of G acts through computable maps. This class of actions is intimately related to other natural classes of group actions, such as subshifts of finite type, see [8, 3]. The understanding of this relationship has led to interesting results, such as the classification of the numbers that arise as topological entropies of subshifts of finite type [9] in $\mathbb{Z}^2$ and the existence of chaotic behavior at zero-temperature for locally constant potentials [6].

While the class of EDS satisfies several interesting dynamical properties, in general it is not closed under topological factor maps (see the examples in [3, Propositions 8.1 and 8.2]). However, the class of topological factors of an EDS does satisfy a weak computability constraint: it must be topologically conjugate to the inverse limit of a (non-necessarily uniform) sequence of effective subshifts [3, Corollary 8.8]. This led the authors of [3] to define the class of *weakly effective dynamical systems* (wEDS) as that of the groups actions $G \curvearrowright X$ which are topologically conjugate to the inverse limit of a sequence of effective G-subshifts. The class of wEDS is stable under topological factor maps, and every EDS is a wEDS.

The main motivation behind the name wEDS in [3] was an initial belief that they might characterize the class of topological factors of EDS ([3, Question 8.12]). Our main result is that in fact there exists wEDS which are not the topological factor of any EDS in a large class of groups.

Theorem A. *For any finitely generated group with decidable word problem and undecidable domino problem, there exists a wEDS which is not the topological factor of any EDS.*

The domino problem of a finitely generated group G is undecidable if there is no algorithm which, given as input a finite set of forbidden patterns, can decide whether the subshift of finite type induced by those patterns is nonempty. This notion is based on the classical result by Berger [4] that states, in modern terms, that the domino problem is undecidable for $G = \mathbb{Z}^2$. We refer the reader to [1] for a survey on the domino problem for groups, and note that it has been conjectured that the only groups for which the domino problem is decidable are the virtually free groups [2].

The proof of Theorem A requires two ingredients which, from our point of view, are interesting results by themselves. These results deal with the notion of universality for topological factors. Let us fix a group G. We say that an EDS $G \curvearrowright \mathfrak{U}$ is *universal* if every other EDS of the group G can be

obtained as a topological factor of $G \curvearrowright \mathfrak{U}$. Similarly, a wEDS of G is called *universal* is every other wEDS of G is a topological factor of it.

The first a ingredient is a characterization, initially proved for $\mathbb{Z}^d$-actions by Hochman [7], of the groups with decidable word problem which admit a universal EDS.

Theorem B. *Let G be a finitely generated group with decidable word problem. There exists a universal EDS $G \curvearrowright \mathfrak{U}$ if and only if the domino problem of G is decidable.*

We remark that Theorem B is just the natural extension of Theorems 1.1 and 1.2 of [7] to finitely generated groups with decidable word problem, and that this generalization does not present any additional difficulties. We provide this proof in Section 3 for the sake of completeness.

The second ingredient is the existence of universal wEDS on any finitely generated group.

Theorem C. *For every finitely generated group G there exists a universal wEDS $G \curvearrowright \mathfrak{X}$.*

We will now show that Theorem A follows from Theorem B and Theorem C. Suppose that every wEDS on G is the topological factor of some EDS in G, it follows from Theorem C that there exists a universal EDS on G. Indeed, by our assumption the universal wEDS $G \curvearrowright \mathfrak{X}$ admits an EDS extension $G \curvearrowright \mathfrak{U}$. As every wEDS (and in particular every EDS) is the topological factor of $G \curvearrowright \mathfrak{X}$, it follows that $G \curvearrowright \mathfrak{U}$ is universal.

In particular, if G is a finitely generated group with decidable word problem and undecidable domino problem, the conclusion from the argument above would contradict Theorem B, and thus we obtain Theorem A.

In Section 2 we introduce the reader to the the necessary background material and set the notation for the rest of the article. In Section 3 we provide the proof of Propositions 3.1 and 3.2 which together prove Theorem B. In Section 4 we show the existence of a universal wEDS for any finitely generated group with decidable word problem, and then explain how this result is valid even without the assumption of decidable word problem. Finally, in Section 5 we discuss the missing case of groups with decidable domino problem and give a few pointers towards a characterization of topological factors of EDS.

Acknowledgments

2 Preliminaries

We denote the set of non-negative integers by $\mathbb{N}$. For a finite set A we write $A^* = \bigcup_{n \in \mathbb{N}} A^n$ for the set of all words on A. A subset $L \subset A^*$ is called a *language*. We say L is *recursively enumerable* if there is an algorithm which halts on input $w \in A^*$ if and only if $w \in L$. We say that L is *decidable* if both L and $A^* \setminus L$ are recursively enumerable. A partial map $g \colon A^* \to B^*$ is computable if there is an algorithm which on input $(u, v) \in A^* \times B^*$ halts if and only if $v = g(u)$.

Through the canonical binary coding we identify non-negative integers with words in $\{0, 1\}^*$ and thus extend the notions of recursive enumerable and decidable to sets of non-negative integers.

A sequence of recursively enumerable languages $(L_n)_{n \in \mathbb{N}}$ is *uniform* if there is an algorithm with in input $n \in \mathbb{N}$ and a word w halts if and only if $w \in L_n$.

For a finitely generated group G and a finite set of generators $S \subset G$, the *word problem* is the language

$$\mathtt{WP}_S(G) = \{w \in S^* : w \text{ represents the identity in } G\}.$$

We say that a group has *decidable word problem* if $\mathtt{WP}_S(G)$ is a decidable language. We note that this is a property of the group, in the sense that is does not depend upon the set of generators. In most of the article (with the only exception being the discussion after Theorem 4.1) we will consider finitely generated groups with decidable word problem.

We remark that if G is a group with decidable word problem, then there exists a bijection $\nu \colon \mathbb{N} \to G$ with the property that the maps $(n, m) \mapsto \nu^{-1}(\nu(n) \cdot \nu(m))$ and $n \mapsto \nu^{-1}(\nu(n)^{-1})$ are computable.

Consequently, we can speak about recursively enumerable and decidable subsets of G through this identification and perform the group operations algorithmically.

2.1 Subshifts and morphisms

Given a group G acting on topological spaces X and Y by homeomorphisms, we say that a map $f\colon X \to Y$ is a *morphism* if it is continuous and G-equivariant, that is, for every $g \in G$ and $x \in X$ we have $f(gx) = gf(x)$. We say that a morphism f is a *topological factor map* if it is surjective, and we say it is a *topological conjugacy* if it is a homeomorphism.

Given a topological space X and a group G, we endow the space X^G with the product topology and the left shift map given by

$$(g \cdot x)(h) = x(g^{-1}h) \text{ for all } x \in X^G, g, h \in G.$$

Definition 2.1. A subshift is a closed and G-invariant subset of A^G, where A is a finite set endowed with the discrete topology.

Fix a finite set A. Given a finite $F \subset G$, a map $p\colon F \to A$ is called a *pattern* with *support* F and its associated cylinder set is given by $[p] = \{x \in A^G : x|_F = p\}$. A subshift can be equivalently defined as a subset of A^G which avoids a collection of patterns. That is, for which there exists a collection $\mathcal{F}$ of patterns such that

$$X = X_{\mathcal{F}} = A^G \setminus \bigcup_{g \in G} \bigcup_{p \in \mathcal{F}} g \cdot [p].$$

We shall need the following characterization of morphisms between subshifts, for a proof see [5, Theorem 1.8.1].

Theorem 2.2 (Curtis-Hedlund-Lyndon)**.** *Let $X \subset A^G$ and $Y \subset B^G$ be subshifts and $f\colon X \to Y$ a morphism. There exists a finite $F \subset G$ and $\Phi\colon A^F \to B$ such that*

$$f(x)(g) = \Phi\big((g^{-1} \cdot x)|_F\big) \text{ for all } x \in X, g \in G.$$

2.2 The domino problem

For this short subsection, fix a finitely generated group G with decidable word problem. A subshift $X \subset A^G$ is called of finite type if it there exists a finite set of patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

Definition 2.3. We say that G has decidable domino problem, if there is an algorithm which on input a finite set of patterns $\mathcal{F}$ decides whether the subshift $X_{\mathcal{F}}$ is nonempty.

We note that for every recursively presented and finitely generated group there exists an algorithm which on input a finite set of patterns $\mathcal{F}$ halts if and only if the subshift $X_{\mathcal{F}}$ is empty (see [1, Proposition 9.3.29]). Hence we may alternatively say that G has decidable domino problem, if there is an algorithm which on input a finite set of patterns $\mathcal{F}$ halts if and only if the subshift $X_{\mathcal{F}}$ is nonempty.

We also note that to our current knowledge, the only infinite groups known to have decidable domino problem are the virtually free groups. It has been conjectured that they are the only ones [2].

2.3 Computability in zero-dimensional spaces

Given a finite set A, we consider the Cantor space $A^{\mathbb{N}}$ endowed with the product of the discrete topology. For a word $w \in A^*$ we denote by $[w]$ the cylinder set of all $x \in A^{\mathbb{N}}$ which begin with w.

Next we shall succinctly introduce computability notions for subsets of the Cantor space. For a friendlier presentation we refer the reader to [3].

Definition 2.4. Let A be a finite set. We say a set $X \subset A^{\mathbb{N}}$ is *effectively closed* if there exists a recursively enumerable language $L \subset A^*$ such that

$$X = A^{\mathbb{N}} \setminus \bigcup_{w \in L} [w].$$

Definition 2.5. Let A, B be finite sets. Given $X \subset A^{\mathbb{N}}$, we say a map $f \colon X \to B^{\mathbb{N}}$ is *computable* if there exists a partial computable map $g \colon A^* \to B^*$ such that for any $x \in X$, $g(x|_{\{0,\ldots,n\}})$ is defined for every n, the cylinders $[g(x|_{\{0,\ldots,n\}})]$ form a nested sequence and

$$\{f(x)\} = \bigcap_{n \in \mathbb{N}} [g(x|_{\{0,\ldots,n\}})].$$

Definition 2.6. Let G be a finitely generated group. We say a left action $G \curvearrowright X$ is a zero-dimensional *effective dynamical system* (EDS) if it is topologically conjugate to a left action $G \curvearrowright Y$ where $Y \subset \{0,1\}^{\mathbb{N}}$ is an effectively closed set and G acts by computable maps.

We note that there is a more general definition of EDS on which G is allowed to act in more general metric spaces (see [3, Definition 3.25]), however, the definitions are equivalent in the zero-dimensional case [3, Proposition 4.10], and furthermore, for recursively presented groups (in particular, for groups with decidable word problem), every EDS is the topological factor of a zero-dimensional EDS [3, Theorem A]. Therefore, for the purpose of this note, the zero-dimensional definition given above is enough.

Definition 2.7. A subshift $X \subset A^G$ is called *effective* if the left shift action $G \curvearrowright X$ is an EDS.

We note that in the case where G is a finitely generated group with decidable word problem, we may identify patterns $p \colon F \to A$ with words in A^* and thus speak about recursively enumerable sets of patterns. The following characterization of effective subshifts is well-known, a proof can be found in [3, Corollary 7.7].

Proposition 2.8. *Let G be a finitely generated group with decidable word problem. A subshift $X \subset A^G$ is effective if and only if there exists a recursively enumerable set of forbidden patterns $\mathcal{F}$ such that*

$$X = A^G \setminus \bigcup_{p \in \mathcal{F}} g[p].$$

A similar description can be given for general zero-dimensional EDS, albeit, with an infinite alphabet. Again, if G is a finitely generated group with decidable word problem, the set $G \times \mathbb{N}$ can be identified by a computable bijection with $\mathbb{N}$, and thus we may speak about effectively closed subsets of $A^{G \times \mathbb{N}}$. We endow the space $A^{G \times \mathbb{N}}$ with the left shift action given by

$$(g \cdot y)(h, n) = y(g^{-1}h, n) \text{ for all } y \in Y, g, h \in G \text{ and } n \in \mathbb{N}.$$

Notice that the left shift action $G \curvearrowright A^{G \times \mathbb{N}}$ is computable. The following characterization of zero-dimensional EDS is quite useful as it encodes the computability of the action in the space. We give a brief proof sketch.

Proposition 2.9. *Let G be a finitely generated group with decidable word problem. An action $G \curvearrowright X$ is a zero-dimensional EDS if and only if it is topologically conjugate to the shift action $G \curvearrowright Y$ for some effectively closed and G-invariant $Y \subset \{0,1\}^{G \times \mathbb{N}}$.*

Proof. The left shift action is computable in $\{0,1\}^{G \times \mathbb{N}}$ and thus if $G \curvearrowright X$ is topologically conjugate to the left shift action $G \curvearrowright Y$ for some effectively closed $Y \subset \{0,1\}^{G \times \mathbb{N}}$, it follows that it is an EDS.

Conversely, if $G \curvearrowright X$ is an EDS, it is topologically conjugate to a left action $G \curvearrowright Z$ where $Z \subset \{0,1\}^{\mathbb{N}}$ and G acts by computable maps. Consider the orbit space $Y \subset \{0,1\}^{G \times \mathbb{N}}$, where $y \in Y$ if and only if $(y(1_G, n))_{n \in \mathbb{N}} \in Z$ and $(y(g, n))_{n \in \mathbb{N}} = g^{-1} \cdot (y(1_G, n))_{n \in \mathbb{N}}$ for all $g \in G$. It is clear that Y is G-invariant and that $G \curvearrowright Z$ is topologically conjugate to the shift action of G on Y. Finally, from the fact that Z is effectively closed and that G acts by computable maps one can deduce that Y is effectively closed. $\square$

2.4 Weak effective dynamical systems and universality

Let $(G \curvearrowright X_n)_{n \in \mathbb{N}}$ be a sequence of actions of a group G on topological spaces X_n by homeomorphisms and a sequence $(\pi_n)_{n \in \mathbb{N}}$ of topological factor maps $\pi_n \colon X_{n+1} \to X_n$. The *inverse limit* associated to these sequences is the space

$$\varprojlim X_n = \{(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n : x_n = \pi_n(x_{n+1}) \text{ for every } n \in \mathbb{N}\},$$

endowed with the pointwise action of G.

Definition 2.10. An action of a group G on a zero-dimensional space is called a *weak effective dynamical system* (wEDS) if it is topologically conjugate to an inverse limit of effective subshifts.

We note that, as a consequence of Proposition 2.9 every zero-dimensional EDS can be represented as an inverse limit of effective subshifts, however, it has the added property that the sequence of effective subshifts $(X_n)_{n\in\mathbb{N}}$ is *uniform*, that is, there is a single algorithm which on input n produces a list of forbidden patterns for X_n. In our definition of wEDS we do not ask for the sequence to be uniform.

Definition 2.11. We say that an EDS (resp. wEDS) $G \curvearrowright \mathfrak{U}$ is *universal* if for every EDS (resp. wEDS) $G \curvearrowright X$ there exists a topological factor map $f \colon \mathfrak{U} \to X$.

3 Existence of universal EDS

The purpose of this section is to prove Theorem B. In Proposition 3.1 we show that a universal EDS exists in any group with decidable domino problem, while in Proposition 3.2 we show that on any group with decidable word problem, the existence of a universal EDS implies the decidability of the domino problem. Theorem B follows from putting both propositions together.

Proposition 3.1. *Let G be a finitely generated group with decidable word and domino problems. There exists a universal EDS $G \curvearrowright \mathfrak{U}$.*

Proof. Let $(L_n)_{n\in\mathbb{N}}$ be a recursive enumeration of all recursively enumerable subsets of patterns of the form $\{0,1\}^F$ for some finite $F \subset G \times \mathbb{N}$. In this way, every zero-dimensional EDS for some $n \in \mathbb{N}$ is topologically conjugate to the shift action on

$$X_n = \{0,1\}^{G\times\mathbb{N}} \setminus \bigcup_{g\in G} \bigcup_{p\in L_n} g[p].$$

Consider a recursive enumeration $p_0, p_1, \ldots$ of all patterns of the form $\{0,1\}^F$ for some finite $F \subset G \times \mathbb{N}$. For each $n \in \mathbb{N}$, we construct a sublanguage $L_n' \subset L_n$ as follows:

For each $k \in \mathbb{N}$, let T_k be the set of patterns $p \in \{p_0, \ldots, p_k\}$ such that the algorithm that recognizes L_n halts on input p on at most k steps. Let $F_k \times I_k$ be the smallest finite subset of $G \times \mathbb{N}$ such that the support of every pattern in T_k is contained in $F_k \times I_k$. Let P_k be the set of all maps $w \colon F_k \to \{0,1\}^{I_k}$ for which there is $p \in T_k$ such that for every $(g,i) \in \text{supp}(p)$ we have $w(g)(i) = p(g,i)$.

Consider the subshift of finite type $Z_{n,k} \subset (\{0,1\}^{I_k})^G$ given by

$$Z_{n,k} = (\{0,1\}^{I_k})^G \setminus \bigcup_{g\in G} \bigcup_{w\in P_k} g[w].$$

We take L_n' as the union of P_k over all k such that the corresponding $Z_{n,k}$ is nonempty.

Notice that there is an algorithm that given the algorithm that recognizes L_n and an integer k, computes the sets T_k, $F_k \times I_k$ and P_k. Furthermore, as the domino problem on G is decidable, there is an algorithm which on finite time decides whether $Z_{n,k}$ is empty, thus it follows that L_n' is recursively enumerable. Moreover, as this algorithm is uniform for all n, we have that $(L_n')_{n\in\mathbb{N}}$ is a uniform sequence of recursively enumerable languages.

Let

$$X_n' = \{0,1\}^{G\times\mathbb{N}} \setminus \bigcup_{g\in G} \bigcup_{p\in L_n'} g[p].$$

By our construction, it follows that X_n' is always nonempty. Furthermore, we have that $L_n = L_n'$ if and only if $X_n \neq \varnothing$, in which case $X_n = X_n'$.

Let $\mathfrak{U} = \prod_{n\in\mathbb{N}} X_n'$ and take $G \curvearrowright \mathfrak{U}$ as the coordinate-wise shift. We have that $\mathfrak{U}$ is nonempty and every G-EDS is the topological factor of $G \curvearrowright \mathfrak{U}$ by a projection map.

Finally, $G \curvearrowright \mathfrak{U}$ is an EDS. To see this, for a pattern $p \in \{0,1\}^{F\times I}$, denote by $\iota_n(p)$ the pattern with support $F \times (I \times \{n\})$ such that $\iota_n(p)(g,(i,n)) = p(g,i)$ for every $g \in F$ and $i \in I$. We have that $G \curvearrowright \mathfrak{U}$ is topologically conjugate to the shift action on

$$\mathfrak{X} = \{0,1\}^{G\times\mathbb{N}^2} \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{g\in G} \bigcup_{p\in L_n'} g[\iota_n(p)].$$

Where the conjugacy $\phi \colon \mathfrak{U} \to \mathfrak{X}$ is given by $\phi((x_n)_{n \in \mathbb{N}})(g, i, j) = (x_j)(g, i)$. As $(L'_n)_{n \in \mathbb{N}}$ is a uniform sequence of recursively enumerable languages, it follows that $\mathfrak{U}$ is effectively closed. $\square$

Proposition 3.2. *Let G be a group with decidable word problem and suppose that there exists a universal EDS $G \curvearrowright \mathfrak{U}$. Then the domino problem of G is decidable.*

Proof. Assume without loss of generality that $\mathfrak{U} \subset \{0, 1\}^{G \times \mathbb{N}}$ and that $G \curvearrowright \mathfrak{U}$ is the left shift map. Let $(u_i)_{i \in \mathbb{N}}$ be a recursively enumerable sequence of maps from a finite subset of $G \times \mathbb{N}$ to $\{0, 1\}$ such that

$$\mathfrak{U} = \{0, 1\}^{G \times \mathbb{N}} \setminus \bigcup_{i \in \mathbb{N}} [u_i].$$

Let $\mathcal{F}$ be a finite set of patterns over the alphabet $A = \{1, \ldots, n\}$ and denote by $X_{\mathcal{F}}$ its corresponding G-SFT. Without loss of generality, by enlarging our finite set of forbidden patterns, we may take a finite subset $W \subset G$ and suppose that every pattern from $\mathcal{F}$ has support W.

Let $(C_1^t, C_2^t, \ldots C_n^t)_{t \in \mathbb{N}}$ be a recursive enumeration of all n-tuples of clopen subsets of $\{0, 1\}^{G \times \mathbb{N}}$ such that $(C_i^t)_{i=1}^n$ is a partition of $\{0, 1\}^{G \times \mathbb{N}}$.

We run the following algorithm which iterates over all tuples $(m, t) \in \mathbb{N} \times \mathbb{N}$:

1. Take $\mathfrak{U}_m = \{0, 1\}^{G \times \mathbb{N}} \setminus \bigcup_{i=1}^m [u_i]$. Construct the set

$$P_{m,t} = \left\{ \varphi \colon W \to A : \mathfrak{U}_m \cap \bigcap_{g \in W} g \cdot C_{\varphi(g)}^t \neq \varnothing \right\}.$$

2. If $P_{m,t} \cap \mathcal{F} = \varnothing$, halt. Otherwise, try the next tuple.

Suppose the algorithm halts, that is, there is (m, t) such that $P_{m,t} \cap \mathcal{F} = \varnothing$. In this case, we can define the map $f \colon \mathfrak{U} \to A^G$ given for $g \in G$ and $x \in \mathfrak{U}$ by the relation

$$f(x)(g) = i \text{ if and only if } x \in g \cdot C_i^t.$$

As $(C_i^t)_{i=1}^n$ is a partition of $\{0, 1\}^{\mathbb{N}}$, the map f is well defined and continuous. Furthermore, if we equip A^G with the shift action, we have that f is G-equivariant, namely, for every $g \in G$ we have that $f(gx) = g(f(x))$. Moreover, as $\mathfrak{U}$ is nonempty, we have that $f(\mathfrak{U})$ is a nonempty G-subshift. Finally, as $\mathfrak{U} \subset \mathfrak{U}_m$ and $P_{m,t} \cap \mathcal{F} = \varnothing$, we obtain that for each $x \in \mathfrak{U}$, we have $f(x)|_W \notin \mathcal{F}$, and thus $f(\mathfrak{U}) \subset X_{\mathcal{F}}$, which is thus nonempty.

Now suppose that $X_{\mathcal{F}}$ is nonempty. As every G-SFT is a G-EDS, it follows that there exists a factor map $f \colon \mathfrak{U} \to X_{\mathcal{F}}$. Consider for each $i \in A$ the partition $(U_i)_{i=1}^n$ given by

$$U_i = \{x \in \mathfrak{U} : f(x)(1_G) = i\}.$$

It follows that these sets are clopen, and thus we can find $t \in \mathbb{N}$ such that $U_i = C_i^t \cap \mathfrak{U}$. By compactness, as $\mathfrak{U} = \bigcap_{m \in \mathbb{N}} \mathfrak{U}_m$, there exists m such that for every $\varphi \colon W \to A$

$$\mathfrak{U}_m \cap \bigcap_{g \in W} g^{-1}[w_{\varphi(g)}^t] = \varnothing \text{ if and only if } \mathfrak{U} \cap \bigcap_{g \in W} g^{-1}[w_{\varphi(g)}^t] = \varnothing.$$

Hence if $\varphi \in P_{m,t}$, we have that for $x \in \mathfrak{U} \cap \bigcap_{g \in W} g^{-1} C_{\varphi(g)}^t = \bigcap_{g \in W} g^{-1} U_i$ we get $f(x)|_W = \varphi$. Furthermore, as $f(x) \in X_{\mathcal{F}}$, we get that $f(x)|_W \notin \mathcal{F}$ and thus we deduce that $P_{m,t} \cap \mathcal{F} = \varnothing$.

We conclude that this algorithm halts if and only if $X_{\mathcal{F}}$ is nonempty and thus the domino problem of G is decidable. $\square$

4 Existence of universal wEDS

Theorem 4.1. *For every finitely generated group with decidable word problem there exists a universal wEDS $G \curvearrowright \mathfrak{X}$.*

Proof. Consider the collection of all tuples of the form (h, s, X, Y, f) where h and s are non-negative integers, X and Y are nonempty effective G-subshifts whose alphabets are finite subsets of $\mathbb{N}$ and f is a topological factor map between X and Y.

There are at most countably many effective G-subshifts (with their alphabets identified with finite subset of $\mathbb{N}$) for any given group G. Moreover, by the Curtis-Hedlund-Lyndon theorem, there can be at most countably many topological morphisms between X and Y and thus at most countably many topological factor maps. Therefore the space of all such tuples is countable.

Notice that this set of tuples is nonempty as the trivial subshift with a single constant configuration is always effective and the identity map is an automorphism. Thus, the set of all such tuples is countably infinite. Let φ be a bijection from $\mathbb{N}$ to the space of all such tuples. Given $n \in \mathbb{N}$, write $\varphi(n) = (h_n, s_n, X_n, Y_n, f_n)$. Consider the space

$$\mathfrak{X}' = \prod_{n \in \mathbb{N}} X_n,$$

It is clear that $\mathfrak{X}'$ can be identified with a compact subset of the Baire space $\mathbb{N}^{\mathbb{N}}$, and furthermore, that G acts naturally on $\mathfrak{X}'$ through the coordinate-wise shift action.

Let $\mathfrak{X} \subset \mathfrak{X}'$ be the set of all $(x_n)_{n \in \mathbb{N}} \in \mathfrak{X}'$ which satisfy for every $n \in \mathbb{N}$ condition (C_n):

$$\text{if } Y_n = X_{s_n} \text{ and } h_n = h_{s_n} + 1, \text{ then } x_{s_n} = f_n(x_n). \tag{C_n}$$

Let us introduce some notation that will be useful in the remainder of the proof. For $n \in \mathbb{N}$, we write $D_0(n) = n$ and define iteratively for $k \geq 1$, $D_k(n) = s_{D_{k-1}(n)}$. We call $D_k(n)$ the k-th descendant of n. We also define the height $H(n)$ as the largest value of k such that for $0 \leq i < k$ we have $h_{D_i(n)} = h_{D_{i+1}(n)} + 1$. We remark that, by vacuity, if $h_n = 0$ then $H(n) = 0$ and also that $H(n) \leq h_n$.

For $k \in \mathbb{N}$, consider the set of integers

$$A_k = \{D_i(n) : n \in \{0, \ldots, k\}, i \in \{0, \ldots, H(n)\}\}.$$

That is, we take all non-negative integers $n \leq k$ and define A_k to be the set of all their descendants up to their height. Notice that each set A_k is finite, and that $(A_k)_{k \in \mathbb{N}}$ forms a monotone increasing sequence of sets with $\bigcup_{k \in \mathbb{N}} A_k = \mathbb{N}$.

Let us argue that $\mathfrak{X}$ is a wEDS. Consider the projection map $\pi_k \colon \mathfrak{X} \to \prod_{n \in A_k} X_n$ given by $\pi_k((x_n)_{n \in \mathbb{N}}) = (x_n)_{n \in A_k}$ and let $\mathfrak{X}_k = \pi_k(\mathfrak{X})$. It is clear that $\mathfrak{X}_k$ is a subshift as it is a closed G-invariant subset of $\prod_{n \in A_k} X_k$. Furthermore, if we define for $i \in \mathbb{N}$ the natural projection map $g_i \colon \mathfrak{X}_{i+1} \to \mathfrak{X}_i$, then $\mathfrak{X}$ is the inverse limit of the sequence $(\mathfrak{X}_k)_{k \in \mathbb{N}}$ endowed with the collection of factor maps g_i. Therefore, in order to show that $\mathfrak{X}$ is a wEDS, we only need to argue that for every $k \geq 0$, the subshift $\mathfrak{X}_k$ is effective. Indeed, let $k \geq 0$ and note first that $\prod_{n \in A_k} X_k$ is a nonempty product of effective subshifts and thus a nonempty effective subshift itself. Furthermore, by our definition of A_k, it follows that $\mathfrak{X}_k$ is precisely the set of elements of $\prod_{n \in A_k} X_k$ which satisfy conditions C_n for each $n \in A_k$. Noting that each of these conditions can be implemented with finitely many forbidden patterns (as a consequence again of the Curtis-Hedlund-Lyndon theorem), we conclude that $\mathfrak{X}_k$ is effective.

Finally, let us show that $G \curvearrowright \mathfrak{X}$ is universal. Let $G \curvearrowright X$ be a wEDS, then it is topologically conjugate to the inverse limit of a sequence of effective G-subshifts. Let $(Y_i)_{i \in \mathbb{N}}$ be said sequence of effective subshifts and for $i \in \mathbb{N}$ let $t_i \colon Y_{i+1} \to Y_i$ be the associated factor maps.

Construct iteratively a sequence $(n_i)_{i \in \mathbb{N}}$ such that $\varphi(n_0) = (0, 0, Y_0, Y_0, \mathrm{Id}_{Y_0})$ and such that for $i \geq 1$ we have $\varphi(n_i) = (i, n_{i-1}, Y_i, Y_{i-1}, t_{i-1})$. Consider the map $\xi \colon \mathfrak{X} \to \prod_{i \in \mathbb{N}} Y_i$ given by

$$\xi((x_n)_{n \in \mathbb{N}}) = (x_{n_i})_{i \in \mathbb{N}}.$$

It is obvious that ξ is continuous and G-equivariant. Moreover, for each $i \in \mathbb{N}$ we have $x_{n_i} \in X_{n_i} = Y_i$. Furthermore, by condition (C_{n_i}) we have that for $i \geq 1$, $t_i(x_{n_i}) = f_{n_i}(x_{n_i}) = x_{n_{i-1}}$, hence the image of any point in $\mathfrak{X}$ lies in the inverse limit of the Y_i. It remains to show that ξ is surjective.

Let $y = (y_i)_{i \in \mathbb{N}}$ such that for $i \in \mathbb{N}$ we have $t_i(y_{i+1}) = y_i$. We will construct a sequence $x = (x_n)_{n \in \mathbb{N}}$ such that $x \in \mathfrak{X}$ and $\xi(x) = y$. To that end, we first set $x_{n_i} = y_i$ for each $i \in \mathbb{N}$. Then iteratively for each $n \in \mathbb{N}$ we do the following: we compute the largest value (if it exists) of $i \in \{0, \ldots, H(n)\}$ for which $x_{D_i(n)}$ is not yet defined.

1. If $x_{D_i(n)}$ is defined for all such i, we do nothing and move on to $n+1$.

2. If $i < H(n)$, we choose $x_{D_i(n)} \in X_{D_i(n)}$ such that $f_{D_i(n)}(x_{D_i(n)}) = x_{D_{i+1}(n)}$ and repeat the process with $i-1$ until all the values $x_{D_i(n)}$ are defined.

3. If $i = H(n)$, then we choose $x_{D_{H(n)}(n)} \in X_{D_{H(n)}(n)}$ arbitrarily and then proceed as above.

Notice that at any step during this process, if x is defined at a coordinate n, then it is also defined in all descendants of n up to $H(n)$. Furthermore, all coordinates which are defined at a given step satisfy condition C_n, and every coordinate is eventually defined. It follows that $x \in \mathfrak{X}$ and clearly $\xi(x) = y$. $\qquad\square$

Let us now consider the situation of a group G which does not have decidable word problem. In this general setting, we may no longer characterize effective subshifts through forbidden patterns on G, but we may identify them in such a way lifting the action of G to a free group [3, Proposition 7.1]. More precisely, if we let $S \subset G$ be a finite set of generators and consider the free group $F(S)$ generated by S and the canonical epimorphism $\phi \colon F(S) \to G$, then a subshift $X \subset A^G$ is effective if and only if its pullback $\widehat{X} \subset A^{F(S)}$ is effective, where

$$\widehat{X} = \{y \in A^{F(S)} : \text{ there exists } x \in X, y(w) = x(\phi(w)) \text{ for all } w \in F_2, \}.$$

Thus in the proof of Theorem 4.1 we may consider instead of G the group $F(S)$ and instead of effective G-subshifts X_n and Y_n we replace them by their pullbacks $\widehat{X}_n$ and $\widehat{Y}_n$. The rest of the proof is exactly the same and at the end we note that the kernel of the action $F(S) \curvearrowright \mathfrak{X}$ contains $\ker(\phi)$ and thus the resulting induced action of G is the universal G-wEDS. We thus obtain Theorem C.

5 Final remarks

Theorem A shows that on finitely generated groups with decidable word problem but undecidable domino problem, the class of wEDS does not characterize the class of topological factors of EDS. Two questions arise naturally.

Question 5.1. Let G be a group with decidable word problem and undecidable domino problem. Is there an additional invariant of recursive nature that characterizes which wEDS are topological factors of EDS?

We remark that if an inverse limit of effective subshifts is uniform (that is, there is a unique algorithm which describes all subshifts in the sequence and factor maps) then it is an EDS, and all zero-dimensional EDS can be represented in this manner [3, Section 4]. In particular, any subsequence of such a uniform sequence gives rise to a wEDS factor and thus all topological factors of EDS arise in this way. However, we do not see any natural way to state this restriction in purely computable terms.

The second question is whether the class of wEDS is the right characterization for topological factors of EDS on groups in which both the word problem and the domino problem are decidable, such as $\mathbb{Z}$.

Question 5.2. Let G be a group with decidable word problem and decidable domino problem. Is every wEDS on G a topological factor of some EDS?

Naturally, if the answer to Question 5.2 is positive, then a universal wEDS would admit a universal EDS for said group as an extension. This hints that if the answer is positive, it should be possible to "effectivize" the construction in the proof of Theorem C using elements from the proof of Proposition 3.1. We were not able to do this, the main difficulty with such an approach is that, at least in principle, one can no longer enumerate all topological factor maps but merely all topological morphisms.

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