Chaos in Bidimensional Models with Short-Range

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Abstract

We construct a short-range potential on a bidimensional full shift and finite alphabet that exhibits a zero-temperature chaotic behaviour as introduced by van Enter and Ruszel. A phenomenon where there exists a sequence of temperatures that converges to zero for which the whole set of equilibrium measures at these given temperatures oscillates between two ground states. Brémont's work shows that the phenomenon of non-convergence does not exist for short-range potentials in dimension one; Leplaideur obtained a different proof for the same fact. Chazottes and Hochman provided the first example of non-convergence in higher dimensions $d \geqslant 3$; we extend their result for $d = 2$ and highlight the importance of two estimates of recursive nature that are crucial for this proof: the relative complexity and the reconstruction function of an extension.

1 Introduction

The states of a system at equilibrium in statistical mechanics are usually described by a family of distributions indexed by an inverse temperature β called Gibbs states. There are several ways in the literature to formalize the notion of Gibbs states; the most common definition in probability and mathematical physics literature is considering DLR equations, originally proposed by R. Dobrushin [13], and by O. Lanford and D. Ruelle [30], this is the standard definition on textbooks of these areas, see [5, 20, 23].

We adopt a more ergodic approach focusing on *equilibrium measures*, which for regular enough potentials correspond to the translation invariant DLR measures, see Ruelle [41], Muir [40] and Keller [27]. A discussion about when the several notions of Gibbsianess are equivalent (or not) can be found in [7, 27, 29, 36, 40, 45].

The existence of an equilibrium measure for a Lipschitz potential on the full shift comes from compactness. In the one-dimensional setting, the equilibrium measure is unique, whereas uniqueness does not necessarily hold in the two-dimensional case. Our main goal is to understand the behavior of the whole set of equilibrium measures as the temperature goes to zero, showing the existence of potentials with a chaotic behaviour in dimension 2. Given a potential φ, for each inverse temperature β consider μ_β an equilibrium state associated with the potential $\beta\varphi$, the existence of weak*-accumulation points for the family of invariant Borel probability measures $(\mu_\beta)_{\beta\geqslant 0}$ as $\beta \to +\infty$ is a trivial consequence of the Banach-Alaoglu theorem. The measures obtained as accumulation points of such families are particular cases of *minimizing measures* (or ground states) that we briefly recall. A minimizing measure μ_{min} is an invariant probability measure that satisfies

$$\int \varphi \, d\mu_{min} = \bar{\varphi} \quad \text{where} \quad \bar{\varphi} := \inf \Big\{ \int \varphi \, d\mu : \mu \text{ translation invariant} \Big\}.$$

The real number $\bar{\varphi}$ is called the *ergodic minimizing value* of the potential φ.

The union of the support of minimizing measures is a compact invariant set, called the *Mather set*, that prescribes the behavior of the equilibrium measures at zero temperature. Many of the ideas in ergodic optimization and the terminology as ergodic minimizing value, minimizing measures and Mather set comes from the theory of Lagrangian dynamics in the continuous setting, see Mather [34], Mañé [33], Fathi [16, 17, 18], and from Aubry-Mather theory in the discrete setting, Forni, Mather [35], Garibaldi, Lopes [21], Garibaldi, Thieullen [22], Su, de la Llave [44], Sorrentino [43]. A thorough review of ergodic optimization is done in Jenkinson [25] in the one-dimensional setting.

For generic Lipschitz potentials, there exists a unique minimizing measure μ_{min} of uniquely ergodic support (see Morris [38] in the one-dimensional setting, easily extendable in the two-dimensional setting). In that case, for any family of equilibrium measures $(\mu_\beta)_{\beta\geqslant 0}$ we have $\mu_\beta \to \mu_{min}$ as $\beta \to +\infty$. On the other hand, if there are at least two minimizing measures, the sequence $(\mu_\beta)_{\beta\geqslant 0}$ might not converges. The notion of zero-temperature chaotic behaviour was introduced by van Enter and Ruszel in the seminal paper [46], see also [2] for a more detailed proof. Nowadays, there are many examples of Lipschitz potentials on dimension one having a non-trivial Mather set that are examples of zero-temperature chaotic behaviour. In the construction of Chazottes and Hochman [9], while the Mather set is highly complex, the potential is geometrically simple as it is obtained as the distance function to the Mather set. In the work of Coronel and Rivera-Letelier [11], the Mather set is quite simple and may be equal to two ergodic measures with disjoint supports, but the potential is less explicit and given by a decreasing sequence of subshifts of finite type with alternating large topological entropy. In the construction of two of us and E. Garibaldi [4], the Mather set is reduced to two fixed points $\{0^\infty\} \cup \{1^\infty\}$, while the potential is a family of Lipschitz functions that are long-range locally constant; a zero-temperature phase diagram is then obtained showing a relationship between zero-temperature chaotic behaviour and cancellation of the Peierls barrier. As far as we know, the understanding of such a rich structure in higher dimension for Lipschitz potentials is out of reach.

In the discussion that follows, we shall restrict ourselves to the class of short-range potentials. In the one-dimensional setting, the Mather set of a short-range potential could have a rich structure of minimizing measures. It is a remarkable result that in this setting, the zero-temperature limit of Gibbs measures always exists and selects a particular minimizing measure that is not necessarily ergodic. This result was originally proven by Brémont [6], and was later given other proofs by Chazottes, Gambaudo and Ugalde [8], Leplaideur [31], and by Garibaldi and Thieullen [22], which also provide an algorithm that identifies the limiting minimizing measure. In the one-dimensional setting, for short-range potentials, the Mather set is reduced to a finite disjoint union of subshifts of finite type (including periodic orbits), and the limiting minimizing measure is some barycenter of the measures of maximal topological entropy of these subshifts. The extension of Brémont's results to a countable alphabet has been undertaken by Jenkinson, Mauldin and Urbánski [26], Morris [37], Kempton [28] for the BIP case, and recently the transitive case in [3].

The status of the zero-temperature chaotic behaviour for short-range potentials in higher dimensions, $d \geqslant 2$, is completely different. Chazottes and Hochman in [9] constructed for every $d \geqslant 3$ an example of a short-range potential exhibiting a zero-temperature chaotic behaviour. The dimension in their result needs to be greater or equal to 3 because the proof relies heavily on a theorem of Hochman [24] which realizes any one-dimensional effectively closed action as the factor of the subaction of a $\mathbb{Z}^3$-subshift of finite type. After this result, the only case missing was $d = 2$. Our main result is an extension of their results for dimension 2.

Theorem 1.1. There exists a finite alphabet $\mathscr{A}$ and a short-range potential φ on a bidimensional full shift that exhibits the phenomenon of zero-temperature chaotic behaviour.

Our construction is based on the simulation theorem of Aubrun and Sablik [1] which states that every one-dimensional effectively closed subshift, extended vertically trivially to a 2-dimensional subshift, is a topological factor of a subshift of finite type of zero topological entropy. We remark that this result was simultaneously proven by Durand, Romashchenko, and Shen [14, 15], which is the main tool used by Chazottes and Shinoda [10].

While quite intricate, the extension constructed by Aubrun and Sablik has the advantage of being quite explicit, whereas the proof by Durand, Romashchenko, and Shen is based on Kleene's fixed point theorem. We use the Aubrun-Sablik construction to produce a few estimates which are not explicit in [9]. These estimates provide bounds that control the relative complexity of the SFT extension. More precisely, we give an explicit bound of the reconstruction function of the extension, thus avoiding the need to use Kleene's fixed point theorem.

The outline of the proof is the following. In the second section, we give the main definitions, outline the strategy's main ideas, and give the proof of the Theorem 1.1 assuming a number of estimates that arise from a yet unspecified construction. In the third section we explain the detailed construction of the one-dimensional subshift. In the fourth section the prove the estimates pertaining the bounds for the topological entropy. In the fifth section we prove the two estimates on the reconstruction function and complexity function in the Aubrun-Sablik simulation theorem.

The present paper is part of the thesis of the third author Gregório Dalle Vedove. A preliminary version was submitted to arxiv [12] at about the same time when a paper of Chazottes and Shinoda [10] was submitted proving the same result but with a different proof.

2 Definitions and outline of the proof

We summarize our setting in the following definitions.

Definition 2.1. Let $\mathscr{A}$ be a finite set called *alphabet* and $d \geqslant 1$ an integer. The space of d-dimensional configurations $\Sigma^d(\mathscr{A}) = \mathscr{A}^{\mathbb{Z}^d}$ is the *d-dimensional full shift*. The *shift action* is the $\mathbb{Z}^d$-action given $\sigma = (\sigma^u)_{u\in\mathbb{Z}^d}$, $\sigma^u \colon \Sigma^d(\mathscr{A}) \to \Sigma^d(\mathscr{A})$ defined by

$$\sigma^u(x) = y \;\text{ if } y(v) = x(u+v) \text{ for every } x, y \in \Sigma^d(\mathscr{A}) \text{ and } v \in \mathbb{Z}^d.$$

We recall that an invariant probability measure μ for the $\mathbb{Z}^d$ action is a Borel measure on $\Sigma^d(\mathscr{A})$ such that for every Borel set B we have $\mu(\sigma^u(B)) = \mu(B)$ for every $u \in \mathbb{Z}^d$.

The set of invariant probability measures is denoted by $\mathscr{M} = \mathscr{M}(\Sigma^d(\mathscr{A}), \sigma)$.

In this article we choose a function $\varphi : \Sigma^2(\mathscr{A}) \to \mathbb{R}$ that is supposed to describe the energy contribution at the origin of the lattice $\mathbb{Z}^d$. A *potential* is a function $\varphi\colon \Sigma^d(\mathscr{A}) \to \mathbb{R}$.

Definition 2.2. Let $\varphi : \Sigma^d(\mathscr{A}) \to \mathbb{R}$ be a Lipschitz function.

1. The *pressure* of the potential φ is the real number

$$P(\varphi) := \sup_{\mu\in\mathscr{M}} \Big\{ h(\mu) - \int \varphi \, d\mu \Big\}.$$

   where $h(\mu)$ denotes the *Kolmogorov-Sinai entropy* of μ (definition 2.20).

2. An *equilibrium measure at inverse temperature* β, is an invariant probability measure μ_β that maximizes the pressure

$$P(\beta\varphi) = h(\mu_\beta) - \int \beta\varphi \, d\mu_\beta.$$

The set of equilibrium measures at inverse temperature β is denoted by $\mathscr{M}_e(\beta\varphi)$.

The general strategy follows the outlines presented in van Enter and Ruszel [46], Coronel and Rivera-Letelier [11], and Chazottes and Hochman [9].

Definition 2.3. The phenomenon of *zero-temperature chaotic behaviour* holds for a potential $\varphi : \Sigma^d(\mathscr{A}) \to \mathbb{R}$ when there exists a sequence of inverse temperatures $(\beta_k)_{k\geqslant 0}$ going to infinity and two disjoint compact sets $\widetilde{G}_1$ and $\widetilde{G}_2$ of $\Sigma^d(\mathscr{A})$ such that, if for every $k \geqslant 0$, μ_{β_k} is any choice of an ergodic equilibrium measure $\mu_{\beta_k} \in \mathscr{M}_e(\beta_k\varphi)$, the support of any weak*-accumulation point of the odd subsequence $(\mu_{\beta_{2k+1}})_{k\geqslant 0}$, respectively of the even subsequence $(\mu_{\beta_{2k}})_{k\geqslant 0}$, is included in $\widetilde{G}_1$, respectively in $\widetilde{G}_2$.

We shall restrict ourselves to the class of *short-range* potentials.

Definition 2.4. Let $\mathscr{A}$ be a finite set and $D \geqslant 1$ be an integer. A function $\varphi\colon \Sigma^d(\mathscr{A}) \to \mathbb{R}$ is *short-range (of range D)* if

$$x|_{[\![1,D]\!]^d} = y|_{[\![1,D]\!]^d} \;\Rightarrow\; \varphi(x) = \varphi(y) \text{ for every } x, y \in \Sigma^d(\mathscr{A}),$$

where $x|_{[\![1,D]\!]^d}$ denotes the restriction of a configuration x to the square $[\![1, D]\!]^d$.

We recall now several definitions.

Definition 2.5. Let $\mathscr{A}$ be a finite set.

1. If S is a finite subset of $\mathbb{Z}^d$, a *pattern with support S* is a partial configuration $p \in \mathscr{A}^S$. The set $S = \mathrm{supp}(p)$ is called the *support* of p. For $d = 1$ a pattern is called a *word*. If $x \in \Sigma^d(\mathscr{A})$, $p = x|_S$ denotes the pattern obtained by taking the restriction of x to S. A pattern p of size n is a pattern of the form $p \in \mathscr{A}^{[\![1,n]\!]^d}$.

2. The shift action extends to an action over the set of patterns, that is, for every $u \in \mathbb{Z}^d$ and $p \in \mathscr{A}^S$ we say that $\sigma^u(p) = p'$ if and only if $p' \in \mathscr{A}^{S-u}$ and for every $v \in S - u$ we have $p'(v) = p(u + v)$.

3. A pattern $p \in \mathscr{A}^S$ *appears* in a configuration $x \in \Sigma^d(\mathscr{A})$ if there exists $u \in \mathbb{Z}^d$ such that $\sigma^u(x)|_S = p$. We write $p \sqsubset x$. More generally a pattern $p \in \mathscr{A}^S$ *appears* in a pattern $q \in \mathscr{A}^T$ if there exists $u \in \mathbb{Z}^d$ such that $S \subseteq T - u$ and $\sigma^u(q)|_S = p$.

Definition 2.6. A *subshift* X is a closed subset of $\Sigma^d(\mathscr{A})$ which is invariant by the shift action σ.

Subshifts can also be given a convenient combinatorial description by exhibiting a set of forbidden patterns, that is, let $\mathscr{F} \subseteq \bigsqcup_{n\geqslant 1} \mathscr{A}^{[\![1,n]\!]^d}$ and consider the set of all $x \in \Sigma^d(\mathscr{A})$ such that $p \not\sqsubset x$ for every pattern $p \in \mathscr{F}$, then it follows that this set is closed and invariant under the shift action. This motivates the following definition.

Definition 2.7. Let $\mathscr{A}$ be a finite set and $\mathscr{F} \subseteq \bigsqcup_{n\geqslant 1} \mathscr{A}^{[\![1,n]\!]^d}$ be a set of patterns called *the set of forbidden patterns*. A subshift $X \subseteq \Sigma^d(\mathscr{A})$ is said to be *generated by* $\mathscr{F}$ if for every $x \in X$ and $p \in \mathscr{F}$, p does not appear in x. More formally

$$X = \Sigma^d(\mathscr{A}, \mathscr{F}) := \{x \in \Sigma^d(\mathscr{A}) : \forall\, p \in \mathscr{F},\ p \not\Subset x\}.$$

A subshift $X = \Sigma^d(\mathscr{A}, \mathscr{F})$ is said to be a *subshift of finite type* (or SFT) if the set of forbidden patterns $\mathscr{F}$ is a finite set.

It is clear that every subshift is generated by some set of forbidden patterns, namely, X is generated by the full set of forbidden patterns $\mathscr{F}_X$ where $p \in \mathscr{F}_X$ if and only if p does not appear in any $x \in X$. However, we remark that a fixed subshift X can be generated by different sets of forbidden patterns. The first step of our construction consists in choosing a 1-dimensional subshift $\widetilde{X} \subseteq \Sigma^1(\widetilde{\mathscr{A}})$ in the following way. The alphabet $\widetilde{\mathscr{A}}$ is made of two alphabets $\widetilde{\mathscr{A}} = \widetilde{\mathscr{A}_1} \cup \widetilde{\mathscr{A}_2}$ where

$$\widetilde{\mathscr{A}_1} = \{0, 1\} \quad \text{and} \quad \widetilde{\mathscr{A}_2} = \{0, 2\}.$$

The subshift $\widetilde{X} = \bigcap_{k\geqslant 0} \widetilde{X}_k$ is obtained as the intersection of a decreasing sequence of subshifts $\widetilde{X}_k$ of controlled complexity

$$\widetilde{X}_{k+1} \subseteq \widetilde{X}_k.$$

Each $\widetilde{X}_k$ contains a disjoint union of two subshifts

$$\widetilde{X}_k^A \sqcup \widetilde{X}_k^B \subseteq \widetilde{X}_k.$$

The subshift $\widetilde{X}_k^A$ consists on configurations over the symbols $\{0, 1\}$ and satisfies,

$$\{1^\infty\} \subset \widetilde{X}_{k+1}^A \subseteq \widetilde{X}_k^A \subseteq \Sigma^1(\widetilde{\mathscr{A}_1}), \quad \widetilde{X}^A := \bigcap_{k\geqslant 0} \widetilde{X}_k^A.$$

The subshift $\widetilde{X}_k^B$ consists on configurations over the symbols $\{0, 2\}$ and satisfies,

$$\{2^\infty\} \subset \widetilde{X}_{k+1}^B \subseteq \widetilde{X}_k^B \subseteq \Sigma^1(\widetilde{\mathscr{A}_2}), \quad \widetilde{X}^B := \bigcap_{k\geqslant 0} \widetilde{X}_k^B.$$

The two subshifts $\widetilde{X}_k^A$ and $\widetilde{X}_k^B$ are chosen so that their relative complexity alternates depending on whether k is odd or even. We use the symbol $0 \in \widetilde{\mathscr{A}}$ to measure the complexity or the frequency of 0 in each word of $\widetilde{X}_k^A$ or $\widetilde{X}_k^B$. We finally make sure that $\widetilde{X}$ is effectively closed as in the following definition. In what follows, we use the definition of *Turing machine* as in Sipser [42, Definition 3.3].

Definition 2.8. A subshift $\widetilde{X} \subseteq \Sigma^1(\widetilde{\mathscr{A}})$ is said to be effectively closed if there exists a set of forbidden words $\widetilde{\mathscr{F}} \subseteq \bigsqcup_{n\geqslant 1} \widetilde{\mathscr{A}}^{[\![1,n]\!]}$ such that $\widetilde{X} = \Sigma^1(\widetilde{\mathscr{A}}, \widetilde{\mathscr{F}})$ and $\widetilde{\mathscr{F}}$ is enumerated by a Turing machine $\widetilde{\mathbb{M}}$. The *time enumeration function* $T^{\widetilde{X}} : \mathbb{N}_* \to \mathbb{N}_*$ associated to $\widetilde{\mathbb{M}}$ is given for $n \geqslant 1$ as the smallest positive integer $T^{\widetilde{X}}(n)$ such that $\widetilde{\mathbb{M}}$ halts on every word of $\widetilde{\mathscr{F}}$ of size at most n in at most $T^{\widetilde{X}}(n)$ steps.

In a second step of the construction we use the Aubrun-Sablik simulation theorem. Their result states that for every effectively closed subshift, one can find a two-dimensional subshift of finite type whose restriction to a one-dimensional subaction is a topological extension of the original effectively closed subshift. More precisely

Theorem 2.9 (Aubrun-Sablik [1])**.** Let $\widetilde{\mathscr{A}}$ be a finite set and $\widetilde{X} = \Sigma^1(\widetilde{\mathscr{A}}, \widetilde{\mathscr{F}})$ be an effectively closed subshift. There exists a finite alphabet $\mathscr{B}$, a subshift of finite type of zero topological entropy $\widehat{X} = \Sigma^2(\widehat{\mathscr{A}}, \widehat{\mathscr{F}})$, where $\widehat{\mathscr{A}} = \widetilde{\mathscr{A}} \times \mathscr{B}$, $\widehat{\mathscr{F}} \subseteq \widehat{\mathscr{A}}^{[\![1,D]\!]^2}$ for some integer $D \geqslant 2$, such that if $\hat{\pi} : \widehat{\mathscr{A}} \to \widetilde{\mathscr{A}}$ is the first projection and $\widehat{\Pi} : \Sigma^2(\widehat{\mathscr{A}}) \to \Sigma^2(\widetilde{\mathscr{A}})$ is the 1-block factor map defined component wise by

$$\widehat{\Pi}(x) = (\hat{\pi}(x_u))_{u\in\mathbb{Z}^2} \text{ for every } x = (x_u)_{u\in\mathbb{Z}^2} \in \Sigma^2(\widehat{\mathscr{A}}),$$

then $\widehat{\Pi}(\widehat{X})$ simulates $\widetilde{X}$ in the sense $\widehat{\Pi}(\widehat{X}) = \widetilde{\widetilde{X}}$ where

$$\widetilde{\widetilde{X}} := \big\{\tilde{x} \in \Sigma^2(\widetilde{\mathscr{A}}) : \forall\, v \in \mathbb{Z},\ (\tilde{x}_{(u,v)})_{u\in\mathbb{Z}} = (\tilde{x}_{(u,0)})_{u\in\mathbb{Z}} \in \widetilde{X}\big\}.$$

The subshift $\widehat{X}$ will be called later the *Aubrun-Sablik SFT simulating* $\widetilde{X}$. The subshift $\widetilde{\widetilde{X}}$ will be called later the *vertically aligned subshift replicating* $\widetilde{X}$.

The remarkable fact is that, although the initial set of forbidden words $\widetilde{\mathscr{F}}$ might be countably infinite, by adding different "colors" $\mathscr{B}$ to the initial alphabet $\widetilde{\mathscr{A}}$ and by imposing a finite set $\widehat{\mathscr{F}}$ of forbidden rules on these juxtaposed colors, the new set of configurations $\widehat{X}$ which respects these rules, after applying the projection $\widehat{\Pi}$, describes exactly the set of vertically aligned configurations of $\widetilde{X}$.

Our proof requires two a priori estimates, see inequalities (1) and (2), that were not stated explicitly in [9]. We recall first several definitions.

Definition 2.10. Let $\mathscr{A}$ be a finite alphabet, $\mathscr{F} \subseteq \bigsqcup_{n\geqslant 1} \mathscr{A}^{[\![1,n]\!]^d}$ be a set of forbidden patterns, and $X = \Sigma^d(\mathscr{A}, \mathscr{F})$.

1. A pattern $w \in \mathscr{A}^S$ is said to be *locally $\mathscr{F}$-admissible* if no pattern p of $\mathscr{F}$ appears in w.
2. A pattern $w \in \mathscr{A}^S$ is said to be *globally $\mathscr{F}$-admissible* if w appears in some configuration $x \in \Sigma^d(\mathscr{A}, \mathscr{F})$.

3. The *reconstruction function* of X is the function $R^X \colon \mathbb{N}_* \to \mathbb{N}_*$ such that, if $n \geqslant 1$, then $R^X(n)$ is the smallest integer $R \geqslant n$ such that, for every locally $\mathscr{F}$-admissible pattern $p \in \mathscr{A}^{\llbracket -R,R \rrbracket^d}$, the subpattern $p|_{\llbracket -n,n \rrbracket^d}$ is globally $\mathscr{F}$-admissible.

By a standard compactness argument, it follows that every subshift admits a well defined reconstruction function. Our proof relies on the fact that the reconstruction function of the Aubrun-Sablik SFT increases at most exponentially for a particular choice of the initial 1-dimensional set of forbidden words $\widetilde{\mathscr{F}}$.

The first a priori estimate is given in the Inequality (1) of the following proposition:

Proposition 2.11. Let $\widetilde{\mathscr{A}}$ be a finite set, $\widetilde{\mathscr{F}} = \bigsqcup_{n\geqslant 1} \widetilde{\mathscr{F}}_n$, $\widetilde{\mathscr{F}}_n \subseteq \widetilde{\mathscr{A}}^{\llbracket 1,n \rrbracket}$ be a set of forbidden words enumerated by a Turing machine $\widetilde{\mathbb{M}}$, $T^{\tilde{X}}$ be the time enumeration function associated to $\widetilde{\mathbb{M}}$ (Definition 2.8), and $\widehat{X}$ be the Aubrun-Sablik SFT simulating $\widetilde{X}$ given in 2.9. We assume that $\widetilde{\mathscr{F}}$ satisfies :

1. $\widetilde{\mathbb{M}}$ enumerates all patterns of $\widetilde{\mathscr{F}}$ in increasing order (words of $\widetilde{\mathscr{F}}_n$ are enumerated before those in $\widetilde{\mathscr{F}}_{n+1}$).

2. $R^{\tilde{X}}(n) \leqslant Cn$, for some constant C,

3. $T^{\tilde{X}}(n) \leqslant P(n)|\widetilde{\mathscr{A}}|^n$, for some polynomial $P(n)$.

Then

$$\limsup_{n\to+\infty} \frac{1}{n} \log(R^{\hat{X}}(n)) < +\infty. \tag{1}$$

The second a priori estimate, see Inequality (2) of the next proposition, improves the computation of the complexity of the Aubrun-Sablik extension. It is know that both the vertically aligned subshift $\widetilde{\widetilde{X}}$ and the Aubrun-Sablik SFT $\widehat{X}$ have both zero topological entropy. We actually need a stronger notion of complexity.

We recall first several definitions.

Definition 2.12. Let $X \subseteq \Sigma^d(\mathscr{A})$ be a subshift.

1. The *language of size n of X* is the set of patterns of size n that appear in X

$$\mathscr{L}(X,n) := \Big\{ p \in \mathscr{A}^{\llbracket 1,n \rrbracket^d} : \exists x \in X, \text{ s.t. } p = x|_{\llbracket 1,n \rrbracket^d} \Big\}.$$

2. The *language of X* is the disjoint union of languages of size n

$$\mathscr{L}(X) := \bigsqcup_{n\geqslant 1} \mathscr{L}(X,n).$$

Definition 2.13. Let $\widetilde{X} = \Sigma^1(\widetilde{\mathscr{A}}, \widetilde{\mathscr{F}})$ be an effectively closed subshift, and $\widehat{X} = \Sigma^2(\widehat{\mathscr{A}}, \widehat{\mathscr{F}})$ be the Aubrun-Sablik SFT given in theorem 2.9 that simulates $\widetilde{X}$. The *relative complexity function* of the simulation is the function $C^{\widehat{X}} \colon \mathbb{N}_* \to \mathbb{N}_*$ defined by

$$C^{\widehat{X}}(n) := \sup_{\widetilde{w} \in \mathscr{L}(\widehat{\Pi}(\widehat{X}), n)} \operatorname{card}\big(\{\widehat{w} \in \mathscr{L}(\widehat{X}, n) : \widehat{\Pi}(\widehat{w}) = \widetilde{w}\}\big).$$

By construction the topological entropy of the simulating SFT is zero. This implies that

$$\lim_{n \to +\infty} \frac{1}{n^2} \log(C^{\widehat{X}}(n)) = 0.$$

We prove a stronger result.

Proposition 2.14. Let $\widetilde{X} = \Sigma^1(\widetilde{\mathscr{A}}, \widetilde{\mathscr{F}})$ be an effectively closed subshift as described in proposition 2.11, and $\widehat{X} = \Sigma^2(\widehat{\mathscr{A}}, \widehat{\mathscr{F}})$ be the Aubrun-Sablik SFT given in theorem 2.9 that simulates $\widetilde{X}$. Then

$$\limsup_{n \to +\infty} \frac{1}{n} \log(C^{\widehat{X}}(n)) < +\infty. \tag{2}$$

In the third and last step of the construction we enlarge the alphabet $\widehat{\mathscr{A}}$ by duplicating randomly the symbol 0. Let

$$\widetilde{\widetilde{\mathscr{A}}} = \{0', 0'', 1, 2\} \quad \text{and} \quad \mathscr{A} = \widetilde{\widetilde{\mathscr{A}}} \times \mathscr{B}.$$

Let $\gamma \colon \mathscr{A} \to \widehat{\mathscr{A}}$ be the map that collapses $0'$ and $0''$ to 0, $\Gamma \colon \Sigma^2(\mathscr{A}) \to \Sigma^2(\widehat{\mathscr{A}})$ be the corresponding 1-block factor map defined component-wise and extended to patterns, and

$$\mathscr{F} := \big\{p \in \mathscr{A}^{[\![1,D]\!]^2} : \Gamma(p) \in \widehat{\mathscr{F}}\big\}, \quad X := \Sigma^2(\mathscr{A}, \mathscr{F}) = \big\{x \in \Sigma^2(\mathscr{A}) : \Gamma(x) \in \widehat{X}\big\}. \tag{3}$$

The subshift X will be called thereafter the *duplicating SFT*. The composition maps $\widehat{\pi} \circ \gamma$ and $\widehat{\Pi} \circ \Gamma$ are denoted by

$$\pi := \widehat{\pi} \circ \gamma : \mathscr{A} \to \widetilde{\mathscr{A}} \quad \text{and} \quad \Pi := \widehat{\Pi} \circ \Gamma : \Sigma^2(\mathscr{A}) \to \Sigma^2(\widetilde{\mathscr{A}}). \tag{4}$$

Notice that X is a SFT generated by the finite set of forbidden patterns $\mathscr{F}$.

The short-range potential φ of Theorem 1.1 responsible for the zero-temperature chaotic behaviour phenomenon may now be defined.

Definition 2.15. Let $X = \Sigma^2(\mathscr{A}, \mathscr{F})$ be the duplicating SFT. The short-range potential $\varphi : \Sigma^2(\mathscr{A}) \to \mathbb{R}$ is the function

$$\varphi(x) = \begin{cases} 1 & \text{if} \ \ x|_{[\![1,D]\!]^2} \in \mathscr{F}, \\ 0 & \text{if} \ \ x|_{[\![1,D]\!]^2} \notin \mathscr{F}. \end{cases} \quad \text{for every } x \in \Sigma^2(\mathscr{A}).$$

Notice that $X = \{x \in \Sigma^2(\mathscr{A}) : \varphi \circ \sigma^u(x) = 0 \text{ for every } u \in \mathbb{Z}^2\}$. In particular, the ergodic minimizing value $\bar{\varphi}$ of φ is zero and the Mather set is the support of the set of invariant probability measures supported by X.

$$\bar{\varphi} = 0 \quad \text{and} \quad \text{Mather}(\varphi) \subseteq X.$$

A consequence is that any weak* accumulation point of $(\mu_{\beta\varphi})_{\beta \to +\infty}$ must be a measure supported in X.

The short-range potential is the characteristic function of a cylinder set. We recall several definitions.

Definition 2.16. Let $\mathscr{A}$ be a finite alphabet, $a \in \mathscr{A}$ be a symbol, $S \subseteq \mathbb{Z}^d$ be a subset, $p \in \mathscr{A}^S$ be a pattern of support S, and $P \subseteq \mathscr{A}^S$ be a subset of patterns.

1. The *cylinder generated by* a, denoted by $[a]_0$, is the set of configurations

$$[a]_0 = \{x \in \Sigma^d(\mathscr{A}) : x(0) = a\}.$$

2. The *cylinder generated by* p, denoted by $[p]$, is the set of configurations

$$[p] := \{x \in \Sigma^d(\mathscr{A}) : x|_S = p\}.$$

3. The *cylinder generated by* P, denoted by $[P]$, is the set of configurations

$$[P] := \bigsqcup_{p \in P} [p].$$

The short-range potential is thus the characteristic function of the cylinder of forbidden words $\mathscr{F}$ defined in (3)

$$\varphi = \mathbb{1}_{[\mathscr{F}]} : \Sigma^2(\mathscr{A}) \to \mathbb{R}.$$

Our next task consists in describing the construction of the intermediate subshifts $\widetilde{X}_k$, $\widetilde{X}_k^A$, $\widetilde{X}_k^B$. To this end, we shall introduce the following notations.

Definition 2.17. Let $\mathscr{A}$ be a finite alphabet, and $d \geqslant 1$ be an integer.

1. A *dictionary of size* ℓ *in dimension* d is a subset L of patterns of $\mathscr{A}^{[\![1,\ell]\!]^d}$.

2. The *concatenated subshift* of a dictionary L of size ℓ is the subshift of the form

$$\begin{aligned} \langle L \rangle &= \bigcup_{u \in [\![1,\ell]\!]^d} \bigcap_{v \in \mathbb{Z}^d} \sigma^{-(u+v\ell)}[L], \\ &= \Big\{x \in \Sigma^d(\mathscr{A}) : \exists u \in [\![1,\ell]\!]^d,\ \forall v \in \mathbb{Z}^d,\ (\sigma^{u+\ell v}(x))|_{[\![1,\ell]\!]^d} \in L\Big\}. \end{aligned}$$

We construct by induction two sequences of dictionaries in dimension 1, $(\widetilde{A}_k)_{k\geqslant 0}$ and $(\widetilde{B}_k)_{k\geqslant 0}$ using the alphabets $\widetilde{\mathscr{A}_1}$ and $\widetilde{\mathscr{A}_2}$ respectively in the following way. We choose a sequence of integers $(N_k)_{k\geqslant 0}$, with $N_k \geqslant 4$ and define by induction the size ℓ_k of the dictionaries $\widetilde{A}_k$ and $\widetilde{B}_k$ by $\ell_0 = 2$ and

$$\ell_k = N_k \ell_{k-1}.$$

We assume that each word of $\widetilde{A}_k$ (respectively $\widetilde{B}_k$) is the concatenation of N_k words of $\widetilde{A}_{k-1}$ (respectively $\widetilde{B}_{k-1}$). We define the corresponding concatenated subshifts

$$\widetilde{X}_k^A := \langle \widetilde{A}_k \rangle, \quad \widetilde{X}_k^B := \langle \widetilde{B}_k \rangle.$$

We note $\widetilde{L}_k := \widetilde{A}_k \bigsqcup \widetilde{B}_k$ and assume that the concatenation of two words of $\widetilde{L}_k$ is a subword of the concatenation of two words of $\widetilde{L}_{k+1}$. We define the corresponding concatenated subshift

$$\widetilde{X}_k := \langle \widetilde{L}_k \rangle.$$

Lemma 2.18. Let $\widetilde{\mathscr{A}}$ be a finite alphabet. Let $(N_k)_{k\geqslant 0}$ be a sequence of integers, $N_k \geqslant 4$, $(\ell_k)_{k\geqslant 0}$ be a sequence defined inductively by $\ell_0 = 2$, $\ell_k = N_k \ell_{k-1}$, and $(\widetilde{L}_k)_{k\geqslant 0}$ be a sequence of dictionaries of size $(\ell_k)_{k\geqslant 0}$ in dimension 1 over the alphabet $\widetilde{\mathscr{A}}$. We assume that, for every $k \geqslant 0$, every word in $\widetilde{L}_k$ is the concatenation of N_k words of $\widetilde{L}_{k-1}$, and that the concatenation of two words of $\widetilde{L}_k$ is a subword of the concatenation of two words of $\widetilde{L}_{k+1}$. Let $\widetilde{X} := \bigcap_{k\geqslant 0} \langle \widetilde{L}_k \rangle$. Then

1. $\langle \widetilde{L}_{k+1} \rangle \subseteq \langle \widetilde{L}_k \rangle$ for every $k \geqslant 0$,
2. $\widetilde{X} = \Sigma^1(\mathscr{A}, \widetilde{\mathscr{F}})$ where $\widetilde{\mathscr{F}} := \bigsqcup_{n\geqslant 0} \widetilde{\mathscr{F}}_n$ and $\widetilde{\mathscr{F}}_n$ is the set of words of length n that are not subwords of any concatenation of two words of $\widetilde{L}_k$ for some $\ell_k \geqslant n$,
3. for every $n \geqslant 0$ and $\ell_k \geqslant n$, $\mathscr{L}(\widetilde{X}, n) = \mathscr{L}(\langle \widetilde{L}_k \rangle, n)$. (In other words, a subword of length n of the concatenation of two words of $\widetilde{L}_k$ is globally $\widetilde{\mathscr{F}}$-admissible.)

The previous lemma tells us that $\widetilde{X} = \bigcap_{k\geqslant 0} \widetilde{X}_k$ is generated by the set of forbidden words $\widetilde{\mathscr{F}} := \bigsqcup_{n\geqslant 0} \widetilde{\mathscr{F}}_n$ where $\widetilde{\mathscr{F}}_n$ is the set of words of length n that are not subwords of the concatenation of two words w and w' taken in $\widetilde{A}_k \sqcup \widetilde{B}_k$. A direct consequence of this is that the reconstruction function of $\widetilde{X}$ satisfies

$$R^{\widetilde{X}}(n) \leqslant n \text{ for every } n \geqslant 1.$$

Later on, we will choose a suitable Turing machine $\widetilde{\mathbb{M}}$ which enumerates $\widetilde{\mathscr{F}}$ in such a way that the hypotheses 1 and 3 of proposition 2.11 are satisfied.

Let $\widetilde{\widetilde{\widetilde{X}}}_k, \widetilde{\widetilde{\widetilde{X}}}_k^A, \widetilde{\widetilde{\widetilde{X}}}_k^B$ be the corresponding vertically aligned subshifts

$$\widetilde{\widetilde{\widetilde{X}}}_k := \{\widetilde{x} \in \Sigma^2(\widetilde{\mathscr{A}}) : \forall\, v \in \mathbb{Z},\ (\widetilde{x}_{(u,v)})_{u\in\mathbb{Z}}) = (\widetilde{x}_{(u,0)})_{u\in\mathbb{Z}} \in \langle \widetilde{L}_k \rangle\},$$
$$\widetilde{\widetilde{\widetilde{X}}}_k^A := \{\widetilde{x} \in \widetilde{\widetilde{\widetilde{X}}}_k : \big(\widetilde{x}_{(u,0)}\big)_{u\in\mathbb{Z}} \in \langle \widetilde{A}_k \rangle\}, \quad \widetilde{\widetilde{\widetilde{X}}}_k^B := \{\widetilde{x} \in \widetilde{\widetilde{\widetilde{X}}}_k : \big(\widetilde{x}_{(u,0)}\big)_{u\in\mathbb{Z}} \in \langle \widetilde{B}_k \rangle\}.$$

Then Lemma 2.18 implies that

$$\mathscr{L}(\widetilde{\widetilde{\widetilde{X}}}, n) = \mathscr{L}(\widetilde{\widetilde{\widetilde{X}}}_k, n) \text{ for every } n \geqslant 1 \text{ and } \ell_k \geqslant n.$$

By the simulation theorem, as $\Pi(X) = \widetilde{\widetilde{\widetilde{X}}}$ (Π is defined in Equation (4)), we obtain

$$\Pi(\mathscr{L}(X, n)) = \mathscr{L}(\widetilde{\widetilde{\widetilde{X}}}, n) \text{ for every } n \geqslant 1.$$

Definition 2.19. We denote for $k \geqslant 0$ by L_k, A_k and B_k the dictionaries of size ℓ_k given by,

1. $L_k := \mathscr{L}(X, \ell_k) \subseteq \mathscr{A}^{[\![1,\ell_k]\!]^2}$,
2. $A_k := \big\{p \in L_k : \Pi(p) \in \mathscr{L}(\widetilde{\widetilde{\widetilde{X}}}_k^A, \ell_k)\big\}$,
3. $B_k := \big\{p \in L_k : \Pi(p) \in \mathscr{L}(\widetilde{\widetilde{\widetilde{X}}}_k^B, \ell_k)\big\}$.

We also denote by X_k, X_k^A, x_k^B the corresponding concatenated subshifts

$$X_k := \langle L_k \rangle, \quad X_k^A := \langle A_k \rangle, \quad X_k^B := \langle B_k \rangle.$$

Notice that we obtain a similar structure as in the one-dimensional setting

$$X = \bigcap_{k\geqslant 0} X_k, \quad X_{k+1} \subseteq X_k, \quad X_k^A \cup X_k^B \subseteq X_k, \quad X_{k+1}^A \subseteq X_k^A, \quad X_{k+1}^B \subseteq X_k^B.$$

Contrary to what happens in the case of $\widetilde{X}_k^A, \widetilde{X}_k^B$, the notion of topological entropy will be sufficient to estimate the complexity of the intermediate subshifts X_k^A and X_k^B. The entropy will be evaluated using the frequency of the symbol 0 in the horizontal direction and the duplication of that symbol in the vertical direction. We recall several definitions. See for instance Walters [47] and Keller [27] for further references.

Definition 2.20. Let $\mathscr{A}$ be a finite set and $X \subseteq \Sigma^d(\mathscr{A})$ be a subshift.

1. The *topological entropy* of X is the non negative real number

$$h_{top}(X) := \lim_{n\to+\infty} \frac{1}{n^d} \log \operatorname{card}(\mathscr{L}(X, n)).$$

2. The *canonical generating partition of* $\Sigma^d(\mathscr{A})$ is the partition

$$\mathscr{G} := \{[a]_0 : a \in \mathscr{A}\}.$$

3. The *common refinement* of two partition $\mathscr{P}$ and $\mathscr{Q}$ of $\Sigma^d(\mathscr{A})$ is the partition

$$\mathscr{P} \bigvee \mathscr{Q} := \{P \cap Q : P \in \mathscr{P}, Q \in \mathscr{Q}\}.$$

4. The *dynamical partition of support* $S \subseteq \mathbb{Z}^2$ of a partition $\mathscr{P}$ is the partition

$$\mathscr{P}^S := \bigvee_{u \in S} \sigma^{-u}(\mathscr{P}).$$

5. The *entropy of a finite partition* $\mathscr{P}$ with respect to an invariant measure μ is the quantity

$$H(\mathscr{P}, \mu) := \sum_{P \in \mathscr{P}} -\mu(P) \ln(\mu(P)).$$

6. The *relative entropy* of a finite partition $\mathscr{P}$ given a partition $\mathscr{Q}$ is the non negative real number

$$H(\mathscr{P} \mid \mathscr{Q}, \mu) = \int H(\mathscr{P}, \mu_x^{\mathscr{Q}})\, d\mu(x),$$

   where $\mu_x^{\mathscr{Q}}$ is the conditional measure with respect to $\mathscr{Q}$.

7. The Kolmogorov-Sinai entropy of an invariant measure μ is the quantity

$$h(\mu) := \sup\Big\{ \lim_{n \to +\infty} \frac{1}{n^d} H\big(\mathscr{P}^{[\![1,n]\!]^d}, \mu\big) : \mathscr{P} \text{ is a finite partition of } \Sigma^d(\mathscr{A}) \Big\}.$$

   A measure μ supported in X satisfying $h(\mu) = h_{top}(X)$ is called a *measure of maximal entropy.*

The following proposition is standard (see the references above).

Proposition 2.21. Let $\mathscr{A}$ be a finite set and $X \subseteq \Sigma^d(\mathscr{A})$ be a subshift.

1. There exists an ergodic invariant probability measure μ supported in X such that

$$h_{top}(X) = h(\mu).$$

   Such a measure is called *measure of maximal entropy.*

2. The *Kolmogorov-Sinai entropy* of an invariant probability measure μ satisfies

$$h(\mu) = \lim_{n \to +\infty} \frac{1}{n^d} H(\mathscr{G}^{[\![1,n]\!]^d}, \mu),$$

where $\mathscr{G}$ is the canonical generating partition of item (2) in definition 2.20.

Let $(\beta_k)_{k \geqslant 0}$ be a sequence of inverse temperatures going to infinity. The heart of our proof is a double estimate of the pressure of $\beta_k \varphi$ that prescribes the statistics of the equilibrium measures. At low temperature an equilibrium measure tends to a minimizing measure that maximizes the topological entropy of the Mather set. As $\mathrm{Mather}(\varphi) \subseteq X$ and X is obtained as a decreasing sequence of X_k, each containing two distinguished subshifts X_k^A and X_k^B, we obtain a zero-temperature chaotic behaviour by choosing alternatively

$$\begin{cases} h_{top}(X_k^A) \ll h_{top}(X_k^B) & \text{for } k \text{ even, } k \to +\infty, \\ h_{top}(X_k^B) \ll h_{top}(X_k^A) & \text{for } k \text{ odd, } k \to +\infty. \end{cases}$$

Where $a_k \ll b_k$ for k even, $k \to +\infty$ means that $\lim_{k \to +\infty} \frac{a_{2k}}{b_{2k}} = 0$, analogously for k odd. Because of the duplication process, the topological entropy can be estimated using the frequency of the symbol 0. Let f_k^A (respectively f_k^B) be the largest frequency of the symbol 0 in the words of $\widetilde{A}_k$ (respectively $\widetilde{B}_k$)

$$f_k^A := \max_{p \in \widetilde{A}_k} f_k^A(p), \quad f_k^A(p) := \frac{1}{\ell_k} \mathrm{card}\big(\{i \in [\![1, \ell_k]\!] : p(i) = 0\}\big).$$

Our construction of X_k^A (and X_k^B) will satisfy that

$$\begin{cases} f_k^A \ll f_k^B \ll 1 & \text{for } k \text{ even, } k \to +\infty, \\ f_k^B \ll f_k^A \ll 1 & \text{for } k \text{ odd, } k \to +\infty. \end{cases}$$

We now explain the double estimate for the pressure that are at the heart of the proof: item 3 of Lemma 2.22 and item 2 of Lemma 2.27.

The first estimate is standard

$$P(\beta_k \varphi) = \sup_\mu \{h(\mu) - \beta_k \mu([\mathscr{F}]) \geqslant h(\mu_k^B) - \beta_k \mu_k^B([\mathscr{F}]),$$

where μ_k^B is an ergodic maximal entropy measure of the subshift X_k^B and k is even.

Using the fact that a configuration in the support of μ_k^B is a tilling of square patterns of size ℓ_k that are globally $\mathscr{F}$-admissible, we obtain easily the following estimates.

Lemma 2.22. Let $k \geqslant 0$.

1. For every ergodic probability measure μ satisfying $\mathrm{supp}(\mu) \subseteq X_k^B$
$$\mu([\mathscr{F}]) \leqslant \frac{2D}{\ell_k}.$$

2. The topological entropy of the shift X_k^B is bounded from below by
$$h_{top}\big(X_k^B\big) \geqslant \ln(2) f_k^B.$$

3. The pressure of $\beta_k\varphi$ is bounded from bellow by
$$P(\beta_k\varphi) \geqslant \ln(2) f_k^B - 2D\frac{\beta_k}{\ell_k}.$$

A similar estimate is also valid for X_k^A, f_k^A, instead of X_k^B, f_k^B. The two parameters β_k and ℓ_k will be chosen so that the following first constraint is valid
$$\begin{cases} \frac{\beta_k}{\ell_k} \ll f_k^B, \text{ for } k \text{ even}, k \to +\infty, \\ \frac{\beta_k}{\ell_k} \ll f_k^A, \text{ for } k \text{ odd}, k \to +\infty. \end{cases} \tag{C1}$$

The second estimate is a bound from above of the pressure of $\beta_k\varphi$. In order to obtain it, we need to introduce a sequence of intermediate scales $(\ell'_k)_{k\geqslant 0}$, two intermediate dictionaries $\widetilde{A}'_k$ and $\widetilde{B}'_k$, and assume that $\widetilde{A}_k$ and $\widetilde{B}_k$ are built over $\widetilde{A}'_k$ and $\widetilde{B}'_k$ in the following way.

Definition 2.23. For every $k \geqslant 0$ we define
$$\ell'_k = N'_k \ell_{k-1},$$
where $N'_k \geqslant 2$ is an integer and N_k is a multiple of N'_k with $N_k/N'_k \geqslant 2$. Let $\widetilde{A}'_k$ (respectively $\widetilde{B}'_k$) be some intermediate dictionary over the alphabet $\widetilde{\mathscr{A}}$ of size ℓ'_k
$$\widetilde{A}'_k, \widetilde{B}'_k \subseteq \widetilde{\mathscr{A}}^{[\![1,\ell'_k]\!]},$$
satisfying the property that every word of $\widetilde{A}'_k$ (respectively $\widetilde{B}'_k$) is obtained by concatenating N'_k words of $\widetilde{A}_{k-1}$ (respectively N'_k words of $\widetilde{B}_{k-1}$). We define
$$\widetilde{L}'_k := \widetilde{A}'_k \bigsqcup \widetilde{B}'_k. \tag{5}$$

Assume also that each word of $\widetilde{A}_k$ (respectively $\widetilde{B}_k$) is a concatenation of N_k/N'_k words of $\widetilde{A}'_k$ (respectively $\widetilde{B}'_k$).

We introduce the following notations.

1. $R'_k := 2R^{\widehat{X}}(\ell'_k) + 1$ be *the reconstruction length at the scale* ℓ'_k (see Definition 2.10 for the definition of the reconstruction function $R^{\widehat{X}}$),
2. $M'_k \subseteq \mathscr{A}^{[\![1,R'_k]\!]^2}$ be the set of patterns of size R'_k that are locally $\mathscr{F}$-admissible
$$M'_k := \big\{w \in \mathscr{A}^{[\![1,R'_k]\!]^2} : \forall\, p \in \mathscr{F},\ \forall\, u \in [\![0, R'_k - D]\!]^2,\ p \not\sqsubseteq \sigma^u(w)\big\},$$
(the set M'_k is called *the reconstruction cylinder at scale* ℓ'_k),
3. $T'_k := \big(\lfloor \frac{R'_k}{2} \rfloor - \ell'_k, \lfloor \frac{R'_k}{2} \rfloor - \ell'_k\big) \in \mathbb{Z}^2$ be a translation vector to the center of $[\![1, R'_k]\!]^2$,
4. $Q'_k := T'_k + [\![1, 2\ell'_k]\!]^2 \subseteq \mathbb{Z}^2$ be the central block of indices for which for every $w \in M'_k$, $w|_{Q'_k}$ is globally $\mathscr{F}$-admissible.

The following lemma shows that an ergodic equilibrium measure μ_{β_k} for $\beta_k\varphi$ at low temperature ($\beta_k \gg 1$) has the tendency to give a large mass to sets of configurations that minimize φ, that is, to sets of configurations that are locally $\mathscr{F}$-admissible. Using the trivial estimate

$$0 \leqslant P(\beta_k\varphi) = h(\mu_{\beta_k}) - \int \beta_k\varphi \, d\mu_{\beta_k} \leqslant h_{top}(\Sigma^2(\mathscr{A})) - \beta_k\mu_{\beta_k}([\mathscr{F}]),$$

one proves easily the following bound.

Lemma 2.24. For every k and every ergodic equilibrium measure μ_{β_k},

$$\mu_{\beta_k}\left(\Sigma^2(\mathscr{A})\backslash[M'_k]\right) \leqslant \frac{(R'_k)^2}{\beta_k} \ln(\mathrm{card}(\mathscr{A})) =: \varepsilon_k.$$

The parameter ε_k is supposed to be small at low temperature. Also, ε_k must actually be negligible compared to the frequency f^B_{k-1}. More precisely

$$\varepsilon_k \ll f^B_{k-1}, \text{ for } k \text{ even}, k \to +\infty,$$

and using $H(\varepsilon_k) := -\varepsilon_k \log(\varepsilon_k) - (1 - \varepsilon_k)\log(1 - \varepsilon_k)$, we also need

$$H(\varepsilon_k) \ll f^B_{k-1} \text{ for } k \text{ even}, k \to +\infty.$$

We simplify these two constraints by using $-\varepsilon_k \log(\varepsilon_k) \ll \sqrt{\varepsilon_k}$, and thus, by imposing

$$\begin{cases} \frac{(R'_k)^2}{\beta_k} \ll (f^B_{k-1})^2, \text{ for } k \text{ even}, k \to +\infty, \\ \frac{(R'_k)^2}{\beta_k} \ll (f^A_{k-1})^2, \text{ for } k \text{ odd}, k \to +\infty. \end{cases} \tag{C2}$$

A typical configuration for μ_{β_k} sees the reconstruction cylinder with probability $1-\varepsilon_k$.

Definition 2.25. Consider the space $\Sigma^2(\mathscr{A})$. We define

1. The *canonical base partition* is given by

$$\widetilde{\mathscr{G}} := \{\widetilde{G}_0, \widetilde{G}_1, \widetilde{G}_2\}, \quad \widetilde{G}_{\tilde{a}} := \{x \in \Sigma^2(\mathscr{A}) : \pi(x(0)) = \tilde{a}\}, \ \forall\, \tilde{a} \in \widetilde{\mathscr{A}}.$$

2. The *reconstruction partition at scale* ℓ'_k is the partition $\mathscr{U}_k$ given by

$$\mathscr{U}_k := \{[M'_k], \Sigma^2(\mathscr{A}) \backslash [M'_k]\}.$$

Notice that $\widetilde{\mathscr{G}}$ is a partition of $\Sigma^2(\mathscr{A})$ and not of $\Sigma^1(\widetilde{\mathscr{A}})$. The symbols coming from the simulation theorem are hidden. The only symbols that remain visible are those from the one-dimensional subshift.

An upper bound on the pressure of $\beta_k\varphi$ is given by the entropy of the equilibrium measure

$$P(\beta_k\varphi) \leqslant h(\mu_{\beta_k}).$$

We decompose the computation of the entropy of μ_{β_k} into 3 terms using a standard identity on relative entropies

$$h(\mu_{\beta_k}) = h_{rel}(\mu_{\beta_k}) + \limsup_{n\to+\infty} \frac{1}{n^2}\Big[H\Big(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2} \mid \mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) + H\Big(\mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big)\Big].$$

The first term $h_{rel}(\mu_{\beta_k})$ is the *relative entropy of* μ_{β_k} *at scale* ℓ'_k

$$h_{rel}(\mu_{\beta_k}) := \lim_{n\to+\infty} \frac{1}{n^2} H\Big(\mathscr{G}^{[\![1,n]\!]^2} \mid \widetilde{\mathscr{G}}^{[\![1,n]\!]^2} \bigvee \mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k} \Big).$$

The term $h_{rel}(\mu_{\beta_k})$ is dominant; it computes the entropy of the canonical generating partition $\mathscr{G}$ in $\Sigma^2(\mathscr{A})$, see Definition 2.20, knowing the fact that the $\widetilde{\mathscr{A}}$-symbols are fixed and that large patterns in $\mathscr{A}^{[\![1,n]\!]^2}$ are tiled by almost non overlapping locally admissible patterns of size R'_k. The second term computes the entropy of the canonical base partition knowing the fact that most the time a configuration is vertically aligned. That term is negligible. The last term computes the entropy of a two set partition where one of the sets, the reconstruction cylinder, has large measure $\mu_{\beta_k}([M'_k]) > 1 - \varepsilon_k$. That term is again negligible.

We obtain easily the following estimates.

Lemma 2.26. For every k and every equilibrium measure μ_{β_k},

1. $\displaystyle\limsup_{n\to+\infty} \frac{1}{n^2} H\big(\mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\big) \leqslant H(\varepsilon_k),$

2. $\limsup_{n\to+\infty} \frac{1}{n^2} H\left(\mathscr{G}^{[\![1,n]\!]^2} \mid \mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\right) \leqslant \left(\frac{8}{R'_k} + \varepsilon_k\right) \ln(\mathrm{card}(\widetilde{\mathscr{A}}))$.

The hardest part of the proof is to bound from above the relative entropy of μ_{β_k} at scale ℓ'_k with respect to the complexity of 1-dimensional globally $\widetilde{\mathscr{F}}$-admissible words of length ℓ'_k (a square tile of size R'_k gives in its center a square tile of size ℓ'_k that is globally admissible). Item (1) in the following lemma shows that the frequency of the symbol $0(0' \wedge 0'')$ is dominated by the one of the symbol 2 if k is even. Item (2) is the most difficult estimate to prove. The computation depends on a particular choice of the language $\widetilde{L}'_k$ (see equation 5) with respect to $\widetilde{L}_{k-1}$. If k is even, the frequency of the symbol 0 in words in $\widetilde{B}'_k$ coincides with the frequency f^B_{k-1}, the frequency of 0 in $\widetilde{A}'_k$ is negligible (of the form f^A_{k-1}/N'_k). If μ_{β_k} gives some positive mass to $\widetilde{G}_1$, then the proportion of the space of configurations that can be covered by words in B'_k (words containing only the symbols 0 and 2) is thus less than $\mu_{\beta_k}\big(\Sigma^2(\mathscr{A})\backslash\widetilde{G}_1\big)$. In particular Item (3) gives us the means to show that the support of the measure μ_{β_k} is in $\widetilde{G}_2$ for k even and in $\widetilde{G}_1$ for k odd.

Lemma 2.27. For every k and every equilibrium measure μ_{β_k},

1. $\mu_{\beta_k}(\widetilde{G}_0) \leqslant \frac{2}{N'_k} f^A_{k-1} + (1 - N^{-1}_{k-1})^{-1} f^B_{k-1} + \varepsilon_k$,
2. if k is even, then
$$h_{rel}(\mu_{\beta_k}) \leqslant \Big(\frac{2}{N'_k} f^A_{k-1} + \big(1 - N^{-1}_{k-1}\big)^{-1}\big(\mu_{\beta_k}\big(\Sigma^2(\mathscr{A})\backslash\widetilde{G}_1\big) + \varepsilon_k\big) f^B_{k-1}\Big)\ln(2) + \frac{1}{\ell'_k}\ln(\mathrm{card}(\widetilde{\mathscr{A}})) + \frac{\ln(C'_k)}{{\ell'_k}^2} + \varepsilon_k \ln(2\mathrm{card}(\widehat{\mathscr{A}})),$$
$$P(\beta_k\varphi) \leqslant h_{rel}(\mu_{\beta_k}) + \left(\frac{8}{R'_k} + \varepsilon_k\right)\ln(\mathrm{card}(\widetilde{\mathscr{A}})) + H(\varepsilon_k).$$
3. if k is odd, then the previous estimate is valid with f^A_{k-1} and f^B_{k-1} permuted and $\widetilde{G}_1$ replaced by $\widetilde{G}_2$.

The term $f^B_{k-1}\ln(2)$ in item (2) is the entropy of duplicated and vertically aligned words taken in an intermediate dictionary B'_k of scale ℓ'_k. The term $f^A_{k-1}\ln(2)$ is interpreted similarly. The term $f^A_{k-1}/(f^B_{k-1}N'_k)$ reflects the fact the ratio of the number of 0 between words in A'_k and B'_k is $1/N'_k$ for k even (a word in B'_k contains much more 0 that a word in A'_k). The term $\ln(C'_k)/{\ell'_k}^2$ converges to the entropy of the simulating SFT. Though the entropy of the Aubrun-Sablik subshift has zero entropy, it is not enough to conclude. This argument seems to be missing in the proof in [9]. The purpose of Proposition 2.14

is to give a stronger a priori bound of the growth of the relative complexity function of the simulating SFT provided the set of forbidden patterns $\widetilde{\mathscr{F}}$ are enumerated in a special way. Then, we will use the estimate

$$\frac{1}{\ell'_k} \ll f^B_{k-1}, \text{ for } k \text{ even}, k \to +\infty.$$

In order for $f^B_{k-1}\ln(2)$ to be the dominant term in item (3) of Lemma 2.22 and item (2) of Lemma 2.27, we assume that ℓ'_k and ℓ_k have been chosen according to the following additional constraint: we assume

$$\begin{cases} \frac{f^A_{k-1}}{N'_k} \ll f^B_{k-1}, \text{ for } k \text{ even}, k \to +\infty, \\ \frac{f^B_{k-1}}{N'_k} \ll f^A_{k-1}, \text{ for } k \text{ odd}, k \to +\infty. \end{cases} \tag{C3}$$

Notice that the two conditions (C1) and(C2) give us an interval of temperatures as follows:

$$\ell_k f^B_{k-1} \gg \beta_k \gg \frac{(R'_k)^2}{(f^B_{k-1})^2}, \text{ for } k \text{ even}, k \to +\infty.$$

These two constraints imply an upper bound of N'_k with respect to N_k. Recalling that $\ell'_k = N'_k\ell_{k-1}$, $R'_k \geqslant \ell'_k$ and $\ell_k = N_k\ell_{k-1}$, we have

$$(N'_k)^2 = \left(\frac{\ell'_k}{\ell_{k-1}}\right)^2 \leqslant \left(\frac{R'_k}{\ell_{k-1}}\right)^2 \ll \ell_k\frac{(f^B_{k-1})^3}{\ell^2_{k-1}} = N_k\frac{(f^B_{k-1})^3}{\ell_{k-1}} \ll N_k, \text{ for } k \text{ even}, k \to +\infty,$$

which implies

$$N'_k \ll N_k, \text{ for } k \text{ even}, k \to +\infty.$$

On the other hand, condition (C3) implies a strong constrain on the lower bound of N'_k with respect to N_k as follows

$$\frac{f^A_{k-1}}{N'_k} \ll f^B_{k-1} \ll f^A_{k-1}, \text{ for } k \text{ even}, k \to +\infty,$$

which implies

$$1 \ll N'_k, \text{ for } k \text{ even}, k \to +\infty.$$

As a conclusion, we are forced to choose N'_k satisfying

$$1 \ll N'_k \ll N_k, \text{ for } k \text{ even}, k \to +\infty.$$

The intermediate scale ℓ'_k is fundamental for our proof, since it allows us to choose the sequence $(\beta_k)_k$ used to prove the existence of the chaotic behaviour. This part of the argument is not present in the literature and proofs given by other authors.

We use Proposition 2.11 to obtain an a priori upper bound of the growth of the reconstruction function. This bound is one of the main ingredients of the construction and, it seems not be highlighted in the other papers about the question.The shape of the reconstruction function has logarithmic growth. We don't need the exact growth but an explicit growth to obtain a recursive sequence; see next Definition 3.1. The upper bound depends on two properties of the time enumeration function (Definition 2.8) of the 1-dimensional set of forbidden patterns. More precisely, the time enumeration function must satisfy that the forbidden words are enumerated successively according to their length (and thus the function is non-decreasing), and that the time to enumerate all words of length n is at most polynomial in n.

We construct by induction $\widetilde{\mathscr{F}}_n$, the full set of forbidden words of $\widetilde{X}$ of length n. We define a primary sequence of scales $(\ell_k)_{k\geqslant 0}$ and a intermediate sequence of scales $(\ell'_k)_{k\geqslant 0}$ so that, by choosing first N'_k large enough and $\ell'_k = N'_k\ell_{k-1}$, (C3) is satisfied, by choosing secondly N_k large enough and $\ell_k = N_k\ell_{k-1}$, (C1) and (C2) are satisfied and β_k is chosen. Essentially it all comes down to check that

$$\ell_k(f^B_{k-1})^3 \gg (R'_k)^2, \text{ for } k \text{ even}, k \to +\infty,$$
$$\ell_k(f^A_{k-1})^3 \gg (R'_k)^2 \text{ for } k \text{ odd}, k \to +\infty.$$

As $\widetilde{L}_k$ is constructed by concatenating N_k words of $\widetilde{L}_{k-1}$, it is clear that a Turing machine might be described such that its time enumeration function is at most exponential independently of the choice of N_k. Proposition 2.11 shows that it is enough to choose N_k so that $\ell_k(f^A_{k-1})^3$ is super-exponential in ℓ'_k.

Finally, let us assume that the following constraint holds

$$\begin{cases} f^B_k = f^B_{k-1} & \text{if } k \text{ is even,} \\ f^A_k = f^A_{k-1} & \text{if } k \text{ is odd.} \end{cases} \tag{C4}$$

Then the double estimates, given in Lemma 2.22 item 3 and Lemma 2.27 item 2, can be reduced to the estimate (for k even)

$$\ln(2)f^B_{k-1} + o(f^B_{k-1}) \leqslant P(\beta_k\varphi) \leqslant \mu_{\beta_k}\big(\Sigma^2(\mathscr{A})\backslash\widetilde{G}_1\big)\ln(2)f^B_{k-1} + o(f^B_{k-1}),$$

using the analogous inequalities, which holds for k odd, A and $\widetilde{G}_2$, instead of B and $\widetilde{G}_1$, we obtain

$$\lim_{k\to+\infty} \mu_{\beta_{2k}}(\widetilde{G}_1) = 0, \quad \lim_{k\to+\infty} \mu_{\beta_{2k+1}}(\widetilde{G}_2) = 0.$$

We observe that the two a priori estimates in 2.14 and 2.11 could be simplified drastically. The only property we need is to have an a priori explicit bound (exponential, super-exponential, or more) of the growth of the reconstruction function, and an a priori sub-exponential bound of the growth of the relative complexity function.

We conclude this section by giving the complete proof Theorem 1.1 assuming the estimates in Lemmas 2.22, 2.24, 2.26, 2.27, and assuming the constraints (C1)–(C4).

Proof of Theorem 1.1. Let μ_{β_k} be an equilibrium measure at inverse temperature β_k. Assume k is an even number. Let μ_k^B be the measure of maximal entropy of the concatenated subshift X_k^B. On the one hand, from Lemma 2.22, we have that

$$P(\beta_k\varphi) \geqslant h(\mu_k^B) - \int \beta_k\varphi \, d\mu_k^B \geqslant \ln(2) f_k^B - 2D\frac{\beta_k}{\ell_k}.$$

From the constraints (C1) and (C4), we have that

$$\frac{\beta_k}{\ell_k} \ll f_{k-1}^B = f_k^B, \text{ for } k \text{ even}, \ k \to +\infty, \text{ and}$$

$$P(\beta_k\varphi) \geqslant \ln(2) f_{k-1}^B + o(f_{k-1}^B).$$

On the other hand, by Lemma 2.27

$$\begin{aligned}
h(\mu_{\beta_k}) &= \lim_{n\to+\infty} \frac{1}{n^2} H\Big(\mathscr{G}^{[\![1,n]\!]^2}, \mu_{\beta_k}\Big) \\
&= h_{rel}(\mu_{\beta_k}) + \limsup_{n\to+\infty} \frac{1}{n^2}\Big[H\Big(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2} \mid \mathscr{U}_k^{[\![0,n-R_k']\!]^2}, \mu_{\beta_k}\Big) + H\Big(\mathscr{U}_k^{[\![0,n-R_k']\!]^2}, \mu_{\beta_k}\Big)\Big], \\
P(\beta_k\varphi) &\leqslant \Big(\frac{2}{N_k'} f_{k-1}^A + (1-N_{k-1}^{-1})^{-1}\Big(\mu_{\beta_k}\big(\Sigma^2(\mathscr{A})\backslash\widetilde{G}_1\big) + \varepsilon_k\Big) f_{k-1}^B\Big)\ln(2) \\
&\quad + \frac{1}{\ell_k'}\ln(\mathrm{card}(\widetilde{\mathscr{A}})) + \frac{1}{\ell_k'^{\,2}}\ln(C_k') + \varepsilon_k \ln(2\mathrm{card}(\widehat{\mathscr{A}})) \\
&\quad + \Big(\frac{8}{R_k'} + \varepsilon_k\Big)\ln(\mathrm{card}(\widetilde{\mathscr{A}})) + H(\varepsilon_k).
\end{aligned}$$

Constraints (C2) and (C3) imply

$$\varepsilon_k \ll f_{k-1}^B \ \text{ and } \ H(\varepsilon_k) \ll f_{k-1}^B, \quad \frac{f_{k-1}^A}{N_k'} \ll f_{k-1}^B, \text{ for } k \text{ even}, \ k \to +\infty.$$

The fact that $N_k' \to +\infty$ and $\ell_{k-1} f_{k-1}^B \geqslant 1$ implies

$$\frac{1}{R_k'} \leqslant \frac{1}{\ell_k'} \leqslant \frac{f_{k-1}^B}{N_k'} \ll f_{k-1}^B, \text{ for } k \text{ even}, \ k \to +\infty.$$

Proposition 2.14 implies

$$\limsup_{k\to+\infty} \frac{\ln(C_k')}{\ell_k'} = 0 \ \Rightarrow \ \frac{\ln(C_k')}{\ell_k'^{\,2}} \ll \frac{1}{\ell_k'} = \frac{1}{N_k'\ell_{k-1}} \ll f_{k-1}^B, \text{ for } k \text{ even}, \ k \to +\infty.$$

We finally obtain

$$\ln(2)f^B_{k-1} + o(f^B_{k-1}) \leqslant P(\beta_k\varphi) \leqslant \mu_{\beta_k}\big(\Sigma^2(\mathscr{A})\backslash\widetilde{G}_1\big)\ln(2)f^B_{k-1} + o(f^B_{k-1}),\ \text{for } k \text{ even, and}$$

$$\lim_{k\to+\infty}\mu_{\beta_{2k}}(\widetilde{G}_1) = 0.$$

Finally, using item 1 of Lemma 2.27, $\lim_{k\to+\infty}\mu_{\beta_k}(\widetilde{G}_0) = 0$, we obtain

$$\lim_{k\to+\infty}\mu_{\beta_{2k}}(\widetilde{G}_2) = 1.$$

An analogous argument based on item 3 of Lemma 2.27 yields

$$\lim_{k\to+\infty}\mu_{\beta_{2k+1}}(\widetilde{G}_1) = 1.$$

Which concludes the proof. □

3 The detailed construction

We complete section 2 by providing complete proofs of all the previous statements.

3.1 The one-dimensional effectively closed subshift

We start by defining an iteration process that will generate the language of $\widetilde{X}$ over the alphabet $\widetilde{\mathscr{A}} = \{0,1,2\}$. Recall that we use the symbol "tilde $\sim$" in all the one-dimensional elements. In a first stage, we will define the values $(\ell_k)_{k\geqslant 0}$ recursively and we define the values of the sequence of inverse temperatures $(\beta_k)_{k\geqslant 0}$. In order to work only with integers, instead of the frequencies f^A_k, f^B_k, we shall define the maximum of the number of symbols 0 counted over all words in $\widetilde{A}_k$ and $\widetilde{B}_k$ respectively:

$$\rho^A_k := \ell_k f^A_k, \quad \rho^B_k := \ell_k f^B_k.$$

Definition 3.1 (The recursive sequence)**.**
There exists a partial recursive function $S : \mathbb{N}^4 \to \mathbb{N}^4$

$$(\ell_k, \beta_k, \rho^A_k, \rho^B_k) = S(\ell_{k-1}, \beta_{k-1}, \rho^A_{k-1}, \rho^B_{k-1}).$$

satisfying $\ell_0 = 2$, $\beta_0 = 0$, $\rho^A_0 = \rho^B_0 = 1$ and defined such that the following holds. In the case k is even:

1. $N'_k := \Big\lceil \dfrac{2k\rho^A_{k-1}}{\rho^B_{k-1}} \Big\rceil$, $\ell'_k = N'_k \ell_{k-1}$,

2. $\beta_k := \left\lceil \frac{\ell_{k-1}^2 2^{k\ell'_k}}{(\rho_{k-1}^B)^2} \right\rceil$,

3. $N_k := N'_k \left\lceil \frac{k\beta_k}{N'_k \rho_{k-1}^B} \right\rceil, \ \ell_k = N_k \ell_{k-1}$,

4. $\rho_k^A = 2\rho_{k-1}^A, \ \rho_k^B = N_k \rho_{k-1}^B$,

In the case k is odd: $(\ell_k, \beta_k, \rho_k^A, \rho_k^B)$ are computed as before with A and B permuted.

The previous sequence $(\ell_k, \beta_k, \rho_k^A, \rho_k^B)_{k\geqslant 0}$ has been chosen so that, first the induction step is explicit in terms of simple (computable) operations, and secondly, such that the four constraints (C1)–(C4) are satisfied. We first observe the following inequalities.

Remark 3.2. For all $k \geqslant 1$ we have the following properties:

1. $2k \leqslant N'_k \leqslant 2k\ell_{k-1}$,
2. $2^{k\ell'_k} \leqslant \beta_k \leqslant \frac{\ell_k}{k}$,
3. $N_{k-1} \leqslant N'_k \leqslant N_k$,
4. if k is odd, $\rho_k^A \geqslant \rho_k^B$, if k is even, $\rho_k^B \geqslant \rho_k^A$,
5. $\beta_k \leqslant \frac{\ell_k \beta_{k+1}}{k 2^{(k+1)\ell_k}} \leqslant \beta_{k+1}$,
6. $f_k^A \ll 1, \ f_k^B \ll 1, \ \frac{N_k}{N'_k} \gg 1, \ \frac{N'_k}{N_{k-1}} \gg 1, \ \frac{\beta_{k+1}}{\beta_k} \gg 1$,
7. if k is odd, $f_k^A \gg f_k^B$, if k is even, $f_k^A \ll f_k^B$, when $k \to +\infty$.

Lemma 3.3. The four constraints (C1)–(C4) are satisfied

Proof. We assume that k is even. The constraint (C4) is satisfied thanks to item 4 of 3.1. The constraint (C3) is satisfied thanks to item 1 of 3.1 and

$$\frac{f_{k-1}^A}{f_{k-1}^B} \leqslant \frac{N'_k}{2k} \ll N'_k, \ \text{for } k \text{ even}, \ k \to +\infty.$$

The constraint (C2) is satisfied thanks to item 2 of 3.1 (especially the fact that we chose a superexponential growth $2^{k\ell'_k}$ instead of $2^{\ell'_k}$) and the assumtion on the bound for the reconstruction function (see Proposition 2.11),

$$\limsup_{k\to+\infty} \frac{\ln(R'_k)}{\ell'_k} < +\infty \ \text{and} \ \beta_k \geqslant \frac{2^{k\ell'_k}}{(f_{k-1}^B)^2} \gg \frac{(R'_k)2}{(f_{k-1}^B)}, \ \text{for } k \text{ even}, \ k \to +\infty.$$

The constraint (C1) is satisfied thanks to item 3 of 3.1 and

$$\beta_k \leqslant \frac{N_k \rho_{k-1}^B}{k} \leqslant \frac{\ell_k f_{k-1}^B}{k} \ll \ell_k f_{k-1}^B = \ell_k f_k^B, \text{ for } k \text{ even}, \ k \to +\infty.$$

□

We now construct the effectively closed 1-dimensional subshifts $\widetilde{A}_k$ and $\widetilde{B}_k$

Definition 3.4. For each $k \geqslant 0$ the dictionaries $\widetilde{A}_k$ and $\widetilde{B}_k$ are made of two words of length ℓ_k

$$\widetilde{A}_k = \{a_k, 1^{\ell_k}\} \subset \widetilde{\mathscr{A}_1}^{[\![1,\ell_k]\!]}, \quad \widetilde{B}_k = \{b_k, 2^{\ell_k}\} \subset \widetilde{\mathscr{A}_2}^{[\![1,\ell_k]\!]},$$

defined by induction as follows:

1. $\ell_0 = 2$, $a_0 = 01$ and $b_0 = 02$,

$$\widetilde{A}_0 = \{01, 11\} \quad \text{and} \quad \widetilde{B}_0 = \{02, 22\}.$$

2. if $k \geqslant 1$ is odd

$$a_k = \underbrace{a_{k-1}a_{k-1}\cdots a_{k-1}}_{N_k\text{-times}}, \quad b_k = b_{k-1}2^{(N_k-2)\ell_{k-1}}b_{k-1}, \tag{R1}$$

3. if $k \geqslant 2$ is even

$$a_k = a_{k-1}1^{(N_k-2)\ell_{k-1}}a_{k-1}, \quad b_k = \underbrace{b_{k-1}b_{k-1}\cdots b_{k-1}}_{N_k\text{-times}}. \tag{R2}$$

Notice that by definition, every word in $\widetilde{L}_k = \widetilde{A}_k \bigsqcup \widetilde{B}_k$ is the concatenation of N_k words of $\widetilde{L}_{k-1}$, and that the concatenation of two words of $\widetilde{L}_k$ is a word of $\widetilde{L}_{k+1}$. It follows that the assumptions of Lemma 2.18 are satisfied. We now proceed to prove that lemma. Recall that

$$\widetilde{X} := \bigcap_{k \geqslant 0} \langle \widetilde{L}_k \rangle,$$

and let $\widetilde{\mathscr{F}} := \bigsqcup_{n\in\mathbb{N}} \widetilde{\mathscr{F}}(n)$ be the set of all forbidden patterns which are obtained by taking all words of length n that are not subwords of the concatenation of two words of $\widetilde{L}_k$ for some $k \geqslant 0$ such that $\ell_k \geqslant n$.

Proof of Lemma 2.18.

Proof of item 1. By assumption every word in $\widetilde{L}_{k+1}$ is a concatenation of words in $\widetilde{L}_k$. Then the concatenated subshifts obviously satisfy $\langle \widetilde{L}_{k+1} \rangle \subseteq \langle \widetilde{L}_k \rangle$.

Proof of item 2. Let $x \in \widetilde{X}$ and $n \geqslant 1$. Then $x \in \langle \widetilde{L}_k \rangle$ for some $n \leqslant \ell_k$. Therefore $x \in \Sigma^1(\widetilde{\mathscr{A}}, \widetilde{\mathscr{F}_n})$ and $\widetilde{X} \subseteq \Sigma^1(\widetilde{\mathscr{A}}, \widetilde{\mathscr{F}})$. Conversely let $x \in \Sigma^1(\widetilde{\mathscr{A}}, \widetilde{\mathscr{F}})$ and $k \geqslant 0$. Define the interval

$$I_k := \left[\!\left[1 - \left\lfloor \frac{\ell_k}{2} \right\rfloor, \ell_k - \left\lfloor \frac{\ell_k}{2} \right\rfloor \right]\!\right].$$

For any $j \geqslant k$, as $x \in \Sigma^1(\widetilde{\mathscr{A}}, \widetilde{\mathscr{F}}_{\ell_j})$, $x|_{I_j}$ is a subword of the concatenation of two words of length ℓ_j of $\widetilde{L}_j$. As $\langle \widetilde{L}_j \rangle \subseteq \langle \widetilde{L}_k \rangle$, $x|_{I_j}$ is a subword of the concatenation of words of length ℓ_k of $\widetilde{L}_k$. Let $y_j \in \langle \widetilde{L}_k \rangle$ such that $y_j|_{I_j} = x|_{I_j}$. By compactness of $\langle \widetilde{L}_k \rangle$, the sequence $(y_j)_{j \geqslant 0}$ admits an accumulation point $y = x \in \langle \widetilde{L}_k \rangle$. Therefore $x \in \widetilde{X}$.

Proof of item 3. We have obviously

$$\forall\, n \leqslant \ell_k, \ \mathscr{L}(\widetilde{X}, n) \subseteq \mathscr{L}(\langle \widetilde{L}_k \rangle, n).$$

Conversely consider two words $u_n, v_n \in \widetilde{L}_n$. We want to show that the concatenation $w_n := u_n v_n$ is a subword of some $x \in \widetilde{X}$. We may assume that w_n is a pattern of support $K_n := [\![1 - \ell_n, 2\ell_n - \ell_n]\!]$. We construct by induction a sequence of patterns $(w_m)_{m \geqslant n}$ of support $K_m = [\![a_m, b_m]\!]$, $b_m - a_m = 2\ell_m - 1$, such that

- w_m is equal to the concatenation of two words of L_m,
- $K_m \subseteq K_{m+1}$ and $w_{m+1}|K_m = w_m$,
- if m is even then $b_{m+1} > b_m$, if m is odd then $a_{m+1} < a_m$.

Indeed, assume m is even and w_m has been constructed. Then, by hypothesis on the language $\widetilde{L}_m$, w_m is the subword of the concatenation of two words of $\widetilde{L}_{m+1}$. Let $\widetilde{w}_{m+1} = \widetilde{u}_{m+1} \widetilde{v}_{m+1}$ be the corresponding pattern of support $\widetilde{K}_{m+1} \supseteq K_m$ of length $2\ell_{m+1}$ containing w_m. If $b_{m+1} > b_m$ we choose $K_{m+1} = \widetilde{K}_{m+1}$ and $w_{m+1} = \widetilde{w}_{m+1}$. If $b_{m+1} = b_m$, as $\ell_{m+1} \geqslant 2\ell_m$ ($\ell_{m+1} > \ell_m$ and $\widetilde{L}_{m+1}$ is obtained by concatenating words of $\widetilde{L}_m$), then w_m is a subword of the rightmost word of $\widetilde{w}_{m+1}$, that is $w_n \sqsubset \widetilde{v}_{m+1}$. We choose any word v_{m+1} in $\widetilde{L}_{m+1}$, define the pattern $w_{m+1} = \widetilde{v}_{m+1} v_{m+1}$, and call K_{m+1} the corresponding support. In the case m is odd we do the analogous construction but on the left hand side.

Let x be the configuration such that $x|K_m = w_m$ for every $m \geqslant n$. Let $y_m \in \langle L_m \rangle$ be a configuration such that $w_m = y_m|K_m$. It follows that the sequence $(y_m)_{m \geqslant n}$ admits an accumulation point $y \in \widetilde{X}$ which satisfies $x|K_m = y|K_m$ for every $m \geqslant n$, and therefore $x = y \in \widetilde{X}$. This shows that $w_n \in \mathscr{L}(X, 2\ell_n)$. $\square$

It is clear from the above arguments that $\Sigma^1(\widetilde{A}, \widetilde{\mathscr{F}})$ is an effectively closed subshift. The following lemma shows that $\widetilde{\mathscr{F}}$ satisfies items (1)–(3) of Proposition 2.11.

Lemma 3.5. The following holds:

1. The reconstruction function satisfies $R^{\widetilde{X}}(n) = n$.
2. For every $n \geqslant 0$, there exist unique integers $k \geqslant 1$ and $N_k \geqslant p \geqslant 2$ satisfying
$$\ell_{k-1} < n \leqslant \ell_k \ \text{ and } \ (p-1)\ell_{k-1} < n \leqslant p\ell_{k-1}.$$
If $(N_k - 1)\ell_{k-1} < n \leqslant N_k\ell_{k-1}$, define $\widetilde{\mathscr{F}}'(n) = \widetilde{\mathscr{F}}(n)$. If $n \leqslant (N_k - 1)\ell_{k-1}$, define $\widetilde{\mathscr{F}}'(n)$ as the set of words of length n that are not subwords of any word of the form $\overrightarrow{w_1}\overleftarrow{w_2}$ where $\overrightarrow{w_1}$ is a terminal segment of w_1 of length $(p+1)\ell_{k-1}$, $\overleftarrow{w_2}$ is an initial segment of w_2 of length $(p+1)\ell_{k-1}$, and w_1 or w_2 are either one of the words $a_k, b_k, 1_k, 2_k$. Then
$$\forall\, n \in [\![1, \ell_k]\!],\ \widetilde{\mathscr{F}}'(n) = \widetilde{\mathscr{F}}(n).$$
3. There exists a Turing machine $\widetilde{\mathbb{M}}$ such that the patterns of $\widetilde{\mathscr{F}}$ are enumerated in increasing order, and such that there is a polynomial $P(n)$ such that the time enumeration function satisfies $T^{\widetilde{X}}(n) \leqslant P(n)|\widetilde{\mathscr{A}}|^n$ for every $n \geqslant 0$.

Proof. *Proof of item (1).* A word w of length n that is not in $\widetilde{\mathscr{F}}$ is a subword of the concatenation of two words of length ℓ_k of $\widetilde{L}_k$. Lemma 2.18 shows that $w \in \mathscr{L}(\widetilde{X}, n)$.

Proof of item (2). We may assume $n \leqslant (N_k - 1)\ell_{k-1}$ and $p < N_k$. We have obviously $\widetilde{\mathscr{F}}(n) \subseteq \widetilde{\mathscr{F}}'(n)$. If we assume that k is even, from Definition 3.4 we have that
$$a_k = a_{k-1}(1^{\ell_{k-1}})^{N_k-2}a_{k-1} \quad \text{and} \quad b_k = (b_{k-1})^{N_k}.$$

We set
$$\overleftarrow{a_k}(1) = \overrightarrow{a_k}(1) = a_{k-1}, \quad \overleftarrow{b_k}(1) = \overrightarrow{b_k}(1) = b_{k-1},$$
$$\overleftrightarrow{1_k}(1) = 1_{k-1} := 1^{\ell_{k-1}} \text{ and } \overleftrightarrow{2_k}(1) = 2_{k-1} := 2^{\ell_{k-1}}.$$
Then we define by induction: if $2 \leqslant p < N_k$ then
$$\overleftarrow{a_k}(p) = \overleftarrow{a_k}(p-1)1_{k-1} = a_{k-1}(1_{k-1})^{p-1}$$
and
$$\overrightarrow{a_k}(p) = 1_{k-1}\overrightarrow{a_k}(p-1) = (1_{k-1})^{p-1}a_{k-1},$$
else $\overleftarrow{a_k}(N_k) = \overrightarrow{a_k}(N_k) = a_k$. We also define
$$\overleftarrow{b_k}(p) = \overleftarrow{b_k}(p-1)b_{k-1} = (b_{k-1})^p,$$

$$\overrightarrow{b_k}(p) = b_{k-1}\overrightarrow{b_k}(p-1) = (b_{k-1})^p,$$
$$\overleftrightarrow{1_k}(p) = \overleftrightarrow{1_k}(p-1)1_{k-1} = (1_{k-1})^p$$

and

$$\overleftrightarrow{2_k}(p) = \overleftrightarrow{2_k}(p-1)1_{k-1} = (2_{k-1})^p.$$

If w has length less than $p\ell_{k-1}$ and is a subword of some w_1w_2, say $w_1 = a_k$ and $w_2 = b_k$, by dragging w from the left end point of w_1w_2 to the right end point of w_1w_2, the word w appears successively as a subword of $\overleftarrow{a_k}(p+1)$, $\overleftrightarrow{1_k}(p+1)$, $\overrightarrow{a_k}(p+1)$, $\overrightarrow{a_k}(p+1)\overleftarrow{b_k}(p+1)$, $\overleftarrow{b_k}(p+1)$. A similar reasoning is also true for $w_1 = b_k$ and $w_2 = a_k$. We have shown that $\widetilde{\mathscr{F}}(n) = \widetilde{\mathscr{F}}'(n)$.

Proof of item (3). To compute the time to enumerate successively the words of $\widetilde{\mathscr{F}}(n)$ when $\ell_{k-1} < n \leqslant \ell_k$, we can produce instead an algorithm which enumerates $\widetilde{\mathscr{F}'}$. The time to read/write on the tapes, to update the words $(\overleftarrow{a_k}(p), \overrightarrow{a_k}(p), \overleftarrow{b_k}(p), \overrightarrow{b_k}(p), \overleftrightarrow{1_k}(p), \overleftrightarrow{2_k}(p))$ by adding a word of length ℓ_{k-1}, to concatenate two words $\overrightarrow{w_1}\overleftarrow{w_2}$ from that list, and to check that a given word w of length n is a subword of $\overrightarrow{w_1}\overleftarrow{w_2}$ is polynomial in n. Therefore, the time to enumerate every word up to length n in an alphabet $\widetilde{\mathscr{A}}$ is bounded by $P(n)|\widetilde{\mathscr{A}}|^n$ where $P(n)$ is some fixed polynomial. □

3.2 The intermediate dictionaries

We shall now study the complexity of the set of words $\mathscr{L}(\widetilde{X}, \ell'_k)$ of length $\ell'_k = N'_k\ell_{k-1}$.

Definition 3.6. Let $\widetilde{A}'_k$ and $\widetilde{B}'_k$ be the sub-dictionaries of $\widetilde{A}_k$ and $\widetilde{B}_k$ that are made of subwords of length ℓ'_k that are either initial or terminal words of a word in $\widetilde{A}_k$ and $\widetilde{B}_k$. Formally,

1. if k is odd, $\widetilde{A}'_k = \{a'_k, 1^{\ell'_k}\}$, $\widetilde{B}'_k = \{b'_k, b''_k, 2^{\ell'_k}\}$,

$$a'_k := \underbrace{a_{k-1}\cdots a_{k-1}}_{N'_k \text{ times}}, \quad b'_k := b_{k-1}2^{(N'_k-1)\ell_{k-1}}, \quad b''_k := 2^{(N'_k-1)\ell_{k-1}}b_{k-1}, \tag{R'1}$$

2. if k is even, $\widetilde{A}'_k = \{a'_k, a''_k, 1^{\ell'_k}\}$, $\widetilde{B}'_k = \{b'_k, 2^{\ell'_k}\}$,

$$a'_k := a_{k-1}1^{(N'_k-1)\ell_{k-1}}, a''_k = 1^{(N'_k-1)\ell_{k-1}}a_{k-1} \text{ and } b'_k := \underbrace{b_{k-1}\cdots b_{k-1}}_{N'_k \text{ times}}. \tag{R'2}$$

3. $\widetilde{L}'_k := \widetilde{A}'_k \bigsqcup \widetilde{B}'_k$.

Notice that $\widetilde{A}'_k$ and $\widetilde{B}'_k$ have been chosen so that the words of $\widetilde{A}_k$ (respectively $\widetilde{B}_k$) are obtained by concatenating N_k/N'_k words of $\widetilde{A}'_k$ (respectively $\widetilde{B}'_k$). In particular we have

$$\langle \widetilde{A}_k \rangle \subset \langle \widetilde{A}'_k \rangle \text{ and } \langle \widetilde{B}_k \rangle \subset \langle \widetilde{B}'_k \rangle.$$

We will say that two words $a, b \in \mathscr{A}^\ell$ overlap if there exists a non-trivial shift $0 < s < \ell$ such that the terminal segment of length s of the word a coincides with the initial segment of the word b of the same length, or vice-versa by permuting a and b. Note that we exclude the overlapping where a and b coincide.

The next three results are technical lemmas about the possible types of overlapping of words of A'_k or B'_k. The first lemma asserts that there is no possible overlapping between words of $\widetilde{A}'_k$ and words of $\widetilde{B}'_k$. The next two lemmas characterize the possible overlaps between any two words at each stage k of the iteration process.

Lemma 3.7. In our construction described above, a word from $\widetilde{A}'_k$ and a word from $\widetilde{B}'_k$ do not overlap. Similarly, a word from $\widetilde{A}_k$ and a word from $\widetilde{B}_k$ do not overlap.

Proof. Every word in $\widetilde{A}'_k$ ends with the symbol 1 which does not appear in any word in $\widetilde{B}'_k$. Conversely, every word in $\widetilde{B}'_k$ ends with the symbol 2 that does not appear in any word in $\widetilde{A}'_k$. The same argument is valid for the words in $\widetilde{A}_k$ and $\widetilde{B}_k$. □

The next lemma is formulated for the case k even, but a similar lemma holds for the case k odd. First we need to fix some notations. Consider the rules described in (R2) and (R'2). The *initial segment* of a_k and a'_k, the *terminal segment* of a_k and a''_k, and the *marker* are the following subwords, for k even

$$a_k = \underbrace{a_{k-1}}_{a^I_{k-1}} 1^{(N_k-2)\ell_{k-1}} \underbrace{a_{k-1}}_{a^T_{k-1}}, \quad b'_k = \underbrace{b_{k-1}}_{b^I_{k-1}} b_{k-1}^{(N'_k-2)} \underbrace{b_{k-1}}_{b^T_{k-1}},$$

$$a'_k = \underbrace{a_{k-1}}_{a^I_{k-1}} \underbrace{1^{(N'_k-1)\ell_{k-1}}}_{\text{marker}} \quad a''_k = \underbrace{1^{(N'_k-1)\ell_{k-1}}}_{\text{marker}} \underbrace{a_{k-1}}_{a^T_{k-1}}.$$

Note that $a^I_{k-1} = a^T_{k-1} = a_{k-1}$ and $b^I_{k-1} = b^T_{k-1} = b_{k-1}$.

Lemma 3.8. Let $k \geqslant 1$ be even, $a_k \in \widetilde{A}_k$ and $b_k \in \widetilde{B}_k$ as described in (R2). Then

1. two words of the same type a_k can only overlap on their initial and terminal segment, that is, the segment a^I_{k-1} of one of the two words overlaps the segment a^T_{k-1} of the other word a_k;

2. on the other hand, two words of the same type b_k overlaps itself exactly on a power of b_{k-1} or they have an overlap of length ℓ_{k-2} between b^I_{k-1} and b^T_{k-1}.

Proof. Proof of item (1). Consider a non-trivial shift $0 < s < \ell_k$ and a word $w \in \widetilde{\mathscr{A}}^{[\![1,s+\ell_k]\!]}$ made of two overlapping a_k:

$$a_k = w|_{[\![1,\ell_k[\![}, \quad \widetilde{a}_k := w|_{s+[\![1,\ell_k]\!]}, \quad \forall\, i \in [\![1,\ell_k]\!],\ \widetilde{a}_k(s+i) = a_k(i).$$

We assume first that $0 < s < \ell_{k-1}$. On the one hand a^T_{k-1} of a_k starts with the symbol 0 at the index $i = (N_k-1)\ell_{k-1}+1$. On the other hand the symbol 1 appears in $\widetilde{a}_k$ at the indexes in the range $[\![\widetilde{i},\widetilde{j}]\!] := [\![s+\ell_{k-1}+1, s+(N_k-1)\ell_{k-1}]\!]$. Since $i \in [\![\widetilde{i},\widetilde{j}]\!]$ we obtain a contradiction.

We assume next that $\ell_{k-1} \leqslant s < (N_k-1)\ell_{k-1}$. On the one hand the symbol 1 appears in a_k at the indexes in the range $[\![\widetilde{i},\widetilde{j}]\!] := [\![\ell_{k-1}+1, (N_k-1)\ell_{k-1}]\!]$. On the other hand $\widetilde{a}_k$ starts with the symbol 0 at the index $i = s+1$. We obtain again a contradiction.

We conclude that s should satisfy $s \geqslant (N_k-1)\ell_{k-1}$: two words of the form a_k can only overlap on their initial and terminal segments.

Proof of item (2). We notice that $k-1$ is odd and b_{k-1} has the same structure as a_k in the first item. Two words of the form b_{k-1} only overlap on their initial and terminal segments. Then b_{k-1} cannot be a subword of the concatenation $c = b_{k-1}b_{k-1}$ of two words b_{k-1} unless b_{k-1} coincides with the first or the last b_{k-1} in c. If b_k and $\widetilde{b}_k$ overlap, either $\widetilde{b}_k$ has been shifted by a multiple of ℓ_{k-1}, $s \in \{\ell_{k-1}, 2\ell_{k-1}, \ldots, (N'_k-1)\ell_{k-1}\}$. Note that $k-1$ is an odd number, and so b_{k-1} has the same behavior of a_k described in the previous item. Therefore, it is only possible to have an overlap of a word b_{k-2} of length ℓ_{k-2} between b^T_{k-1} and $\widetilde{b}^I_{k-1}$. □

Lemma 3.9. Let $k \geqslant 1$ be an even integer and a'_k and a''_k as described in (R'2). Then the following holds:

1. two words of the same form a'_k never overlap; the same is true for two words of the same form a''_k;
2. two words a'_k and a''_k overlap if and only if they overlap either partially on their marker or partially on their initial and terminal segments, respectively.

Proof. Proof of item (1). We consider a non-trivial shift $0 < s < \ell'_k$ and two overlapping words of the form a'_k shifted by s. Let $w \in \widetilde{\mathscr{A}}^{[\![1,s+\ell'_k]\!]}$ such that

$$a'_k = w|_{[\![1,\ell'_k]\!]}, \quad \widetilde{a}'_k := w|_{s+[\![1,\ell'_k]\!]}, \quad \forall\, i \in [\![1,\ell'_k]\!],\ \widetilde{a}'_k(s+i) = a'_k(i).$$

We assume first that $\ell_{k-1} \leqslant s < \ell'_k$. On the one hand, $\widetilde{a}'_k$ starts with the symbol 0, $w(s+1) = 0$; on the other hand, $w|_{[\![\ell_{k-1}+1,\ell'_k]\!]}$ contains only the symbol 1. Since $s+1 \in [\![\ell_{k-1}+1, \ell'_k]\!]$ we obtain a contradiction.

We assume next that $0 < s < \ell_{k-1}$. We observe that $k-1$ is odd and the two initial segments a^I_{k-1} of a'_k and $\widetilde{a}'_k$ are of the same form as b_k in the second item. They overlap on a multiple of words of the form a_{k-2} or at their initial and terminal segments. Necessarily $s \geqslant l_{k-2} \geqslant 2$. On the one hand, the initial segment of $\widetilde{a}'_k$ ends with the symbols 01, $w(s+\ell_{k-1}-1)=0$, on the other hand, $w|_{[\![\ell_{k-1}+1,\ell'_k]\!]}$ contains only the symbol 1. Since $s+\ell_{k-1}-1 \in [\![\ell_{k-1}+1,\ell'_k]\!]$ we obtain a contradiction. A similar proof works for a''_k instead of a'_k.

Proof of item (2). We divide our discussion in two cases. Consider first,

$$a'_k = w|_{[\![1,\ell'_k]\!]}, \quad \widetilde{a}''_k := w|_{s+[\![1,\ell'_k]\!]}, \quad \forall\, i \in [\![1,\ell'_k]\!],\ \widetilde{a}''_k(s+i) = a''_k(i).$$

Suppose that $0 < s < \ell_{k-1}$. On the one hand the terminal segment of $\widetilde{a}''_k$ is a word like a_{k-1} and then it starts with the symbol 0 which appears in w at the index $s+(N'_k-1)\ell_{k-1} \in [\![\ell_{k-1},\ell'_k]\!]$. On the other hand $w|_{[\![\ell_{k-1},\ell'_k]\!]}$ contains only the symbol 1. Thus we obtain a contradiction. We conclude that necessarily $\ell_k \leqslant s$ and the two words a'_k and a''_k overlap (partially or completely) on their markers.

We consider next the case,

$$a''_k = w|_{[\![1,\ell'_k]\!]}, \quad \widetilde{a}'_k := w|_{s+[\![1,\ell'_k]\!]}, \quad \forall\, i \in [\![1,\ell'_k]\!],\ \widetilde{a}'_k(s+i) = a'_k(i).$$

Suppose that $0 < s < (N'_k-1)\ell_{k-1}$. On the one hand the initial segment of $\widetilde{a}'_k$ starts with the symbol 0 which is located at the index $s+1 \in [\![1,(N'_k-1)\ell_{k-1}]\!]$ in w. On the other hand $w|_{[\![1,(N'_k-1)\ell_{k-1}]\!]}$ is the marker of a''_k and contains only the symbol 1. We obtain a contradiction. We conclude that it is only possible to have $s \geqslant (N'_k-1)\ell_{k-1}$, which means that the terminal segment of a''_k overlaps with the initial segment of a'_k. Both segments are copies of a_{k-1} and as we are in the case where $k \geqslant 2$ is even, we have that $k-1$ is odd and thus a_{k-1} has the same behavior described in Lemma 3.8 item (2). Therefore the possible overlap can occur (partially or completely) on their initial and terminal segments by the rules described as in Lemma 3.8 item (2). □

3.3 The vertically aligned subshift

We estimate the entropy of the Gibbs measure μ_{β_k} from above in Lemma 2.27 by measuring the frequency of the duplicated symbol $0', 0''$. If k is even most of the symbols 0 are in the words $b_k \in \widetilde{B}_k$. As b_k does not use the symbol 1, a positive frequency of the symbol 1 for a generic μ_{β_k}-configuration tends to decrease the occurrence of the words b_k. The purpose of this section is to quantitatively justify this intuition.

Definition 3.10. Let $\widetilde{\widetilde{A}}{}'_k \subseteq \mathscr{A}^{[\![1,\ell'_k]\!]^2}$ be the bidimensional dictionary of length ℓ'_k of vertically aligned patterns that project onto $\widetilde{L}'_k$, formally defined as

$$\widetilde{\widetilde{A}}{}'_k := \big\{ p \in \mathscr{A}^{[\![1,\ell'_k]\!]^2} : \exists \widetilde{p} \in \widetilde{A}'_k,\ \text{s.t.}\ \ \forall, (i,j) \in [\![1,\ell'_k]\!]^2,\ p(i,j) = \widetilde{p}(i) \big\}.$$

The dictionary $\widetilde{\widetilde{B}}'_k \subseteq \mathscr{A}^{[\![1,\ell'_k]\!]^2}$ is defined analogously. Let $\widetilde{\widetilde{X}}$ be the set of vertically aligned configurations that project onto $\widetilde{X}$

$$\widetilde{\widetilde{X}} := \{x \in \mathscr{A}^{\mathbb{Z}^2} : \exists \widetilde{x} \in \widetilde{X}, x(i,j) = \widetilde{x}(i) \text{ for every } (i,j) \in \mathbb{Z}^2\}.$$

We use the notation $\widetilde{\widetilde{\pi}} \colon \widetilde{\widetilde{X}} \to \widetilde{X}$ or $\widetilde{\widetilde{\pi}} \colon \widetilde{\widetilde{A}}'_k \to \widetilde{A}'_k$ to represent the projection of a vertically aligned configuration or pattern.

Let $p \in \mathscr{A}^{[\![1,n]\!]^2}$ be a large pattern (not necessarily vertically aligned) and consider the set of translates u of small squares of size $2\ell'_k$ inside this pattern p that are vertically aligned and project onto a pattern of $\widetilde{A}'_k$ or $\widetilde{B}'_k$. We introduce the following notations.

Definition 3.11. Let $k \geqslant 2$, $n > 2\ell'_k$, and $p \in \mathscr{A}^{[\![1,n]\!]^2}$. We define

1. $I(p,\ell'_k) := \left\{u \in [\![0, n-2\ell'_k]\!]^2 : \sigma^u(p)|_{[\![1,2\ell'_k]\!]^2} \in \mathscr{L}(\widetilde{\widetilde{X}}, 2\ell'_k)\right\}$,
2. $I^A(p,\ell'_k) := \left\{u \in [\![0, n-\ell'_k]\!]^2 : \sigma^u(p)|_{[\![1,\ell'_k]\!]^2} \in \widetilde{\widetilde{A}}'_k\right\}$,
3. $J^A(p,\ell'_k) := \displaystyle\bigcup_{u \in I^A(p,\ell'_k)} \left(u + [\![1,\ell'_k]\!]^2\right)$.

We define $I^B(p,\ell'_k)$ and $J^B(p,\ell'_k)$ similarly with replacing $\widetilde{\widetilde{A}}'_k$ for $\widetilde{\widetilde{B}}'_k$ in (2) and (3), respectively.

Lemma 3.12. Let $k \geqslant 2$, $n > 2\ell'_k$, $p \in \mathscr{A}^{[\![1,n]\!]^2}$ and $I(p,\ell'_k), J^A(p,\ell'_k), J^B(p,\ell'_k)$ as in Definition 3.11. Let $\tau'_k =: (\ell'_k,\ell'_k) \in \mathbb{N}^2$. Then $J^A(p,\ell'_k) \cap J^B(p,\ell'_k) = \varnothing$ and

$$\tau'_k + I(p,\ell'_k) \subset J^A(p,\ell'_k) \sqcup J^B(p,\ell'_k).$$

Proof. The fact that $J^A(p,\ell'_k)$ and $J^B(p,\ell'_k)$ do not intersect is a consequence of Lemma 3.7. Let $u \in I(p,\ell'_k)$ and $w_* = \sigma^u(p)|_{[\![1,2\ell'_k]\!]^2}$. There exists $w \in \mathscr{L}(\langle \widetilde{L}_k \rangle, 2\ell'_k)$ such that $w_*(i,j) = w(i)$ for all $(i,j) \in [\![1,2\ell'_k]\!]^2$. By definition of $\langle \widetilde{L}_k \rangle$, $w \sqsubset w_1 w_2$ is a subword of the concatenation of two words of $\widetilde{L}_k$. By Definition 3.6 we have that $\langle \widetilde{L}_k \rangle \subseteq \langle \widetilde{L}'_k \rangle$, and also $\mathscr{L}(\langle \widetilde{L}_k \rangle, 2\ell'_k) \subseteq \mathscr{L}(\langle \widetilde{L}'_k \rangle, 2\ell'_k)$.

On the other hand, a word in $\widetilde{L}_k$ is either a word of $\widetilde{A}_k$ or a word of $\widetilde{B}_k$. As $\langle \widetilde{A}_k \rangle \subset \langle \widetilde{A}'_k \rangle$ and $\langle \widetilde{B}_k \rangle \subset \langle \widetilde{B}'_k \rangle$, w_1 and w_2 are obtained as a concatenation of words of $\widetilde{A}'_k$ or $\widetilde{B}'_k$. There exists $0 \leqslant s < \ell'_k$ such that

$$\sigma^s(w)|_{[\![1,\ell'_k]\!]} \in \widetilde{A}'_k \bigsqcup \widetilde{B}'_k.$$

Then

$$u + (s, s) \in I^A(p, \ell'_k) \bigsqcup I^B(p, \ell'_k),$$

and therefore

$$u + \tau'_k \in J^A(p, \ell'_k) \bigsqcup J^B(p, \ell'_k).$$

Which concludes the proof. See Figure 1 for an illustration of this result. □

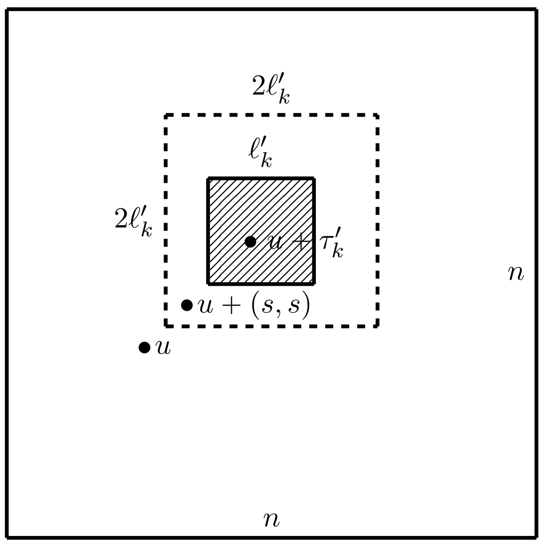

Figure 1: The biggest square is the square pattern $p \in \widetilde{\mathscr{A}}^{[\![0,n]\!]^2}$. Let $u \in I(p, \ell'_k)$. The dashed square of size $2\ell'_k$ is a pattern in $\mathscr{L}(\widetilde{\widetilde{X}}, 2\ell'_k)$. The pattern located in the innermost square of size ℓ'_k belongs to $\widetilde{A}'_k \bigsqcup \widetilde{B}'_k$. The innermost dot represents $u + \tau'_k$.

Lemma 3.13. Let $k \geqslant 2$ be an even integer, $n > 2\ell'_k$, and $p \in \widetilde{\mathscr{A}}^{[\![1,n]\!]^2}$. Let $I^A(p, \ell'_k)$, $J^A(p, \ell'_k)$, $I^B(p, \ell'_k)$, $J^B(p, \ell'_k)$ as in Definition 3.11. Define

$$K^A(p, \ell'_k) = \{v \in J^A(p, \ell'_k) : p(v) = 0\}, \quad K^B(p, \ell'_k) = \{v \in J^B(p, \ell'_k) : p(v) = 0\}.$$

Then

1. $\mathrm{card}(K^B(p, \ell'_k)) \leqslant \left(1 - N_{k-1}^{-1}\right)^{-1} \mathrm{card}(J^B(p, \ell'_k)) f^B_{k-1}$,
2. $\mathrm{card}(K^A(p, \ell'_k)) \leqslant \dfrac{2}{N'_k} \mathrm{card}(J^A(p, \ell'_k)) f^A_{k-1}$.

Proof. Let $k \geqslant 2$ even, $n > 2\ell'_k$ and a fixed $p \in \widetilde{\mathscr{A}}^{[\![1,n]\!]^2}$. To simplify the notations, we write $I^A = I^A(p, \ell'_k)$, $J^A = J^A(p, \ell'_k)$ and so on. As the symbol 0 does not appear in the markers $1^{N'_k \ell_{k-1}} \in \widetilde{A}'_k$ and $2^{N'_k \ell_{k-1}} \in \widetilde{B}'_k$, we only need to consider translates $u \in [\![0, n - \ell'_k]\!]^2$ of I^A (resp. I^B) such that $w_* = \sigma^u(p)|_{[\![1,\ell'_k]\!]^2}$ satisfying $\tilde{\tilde{\pi}}(w_*) \in \{a'_k, a''_k\}$ (resp. $\tilde{\tilde{\pi}}(w_*) = b'_k$).

Item 1. We first enumerate $I^B = \{u_1, u_2, \ldots, u_H\}$. Let $u_h = (u_h^x, u_h^y) \in \mathbb{Z}^2$. Let

$$J^B := \bigcup_{h=1}^{H} J_h \quad \text{where} \quad J_h := u_h + [\![1, \ell'_k]\!]^2, \;\; \tilde{\tilde{\pi}}(\sigma^{u_h}(p))|_{[\![1,\ell'_k]\!]^2} = b'_k,$$

that is, we are only considering the J_h squares of $J^B(p, \ell'_k)$ that contains vertically aligned word b'_k. For each box J_h we divide into N'_k vertical strips of length ℓ_{k-1}. Formally we have

$$J_h = \bigcup_{i=1}^{N'_k} J_{h,i} \quad \text{where} \quad J_{h,i} := u_h + [\![1 + (i-1)\ell_{k-1}, i\ell_{k-1}]\!] \times [\![1, \ell'_k]\!].$$

We construct a partition of J^B inductively by,

$$J^B = \bigsqcup_{h=1}^{H} J_h^*, \quad J_1^* = J_1, \;\; \forall\, h \geqslant 2, \; J_h^* := J_h \backslash (J_1 \cup \cdots \cup J_{h-1}) \,.$$

Let

$$K_h^* := \{v \in J_h^* : p(v) = 0\}, \; K^B := \bigsqcup_{h=1}^{H} K_h^*.$$

It will be enough to show that for every $h \in [\![1, H]\!]$

$$\operatorname{card}(K_h^*) \leqslant \big(1 - N_{k-1}^{-1}\big)^{-1} \operatorname{card}(J_h^*) f_k^B. \tag{6}$$

By definition of u_h, $\tilde{w}_h = \tilde{\tilde{\pi}}(p|_{(u_h + [\![1,\ell'_k]\!]^2)}) = b'_k \in \widetilde{\mathscr{A}}^{\ell'_k}$,

$$\forall\, i, j \in [\![1, \ell_k]\!]^2, \; \tilde{w}_h(u_h^x + i) = b'_k(i).$$

Since b'_k can be decomposed into N'_k subwords of the form b_{k-1}, we denote by $\tilde{w}_{h,i} \in \widetilde{\mathscr{A}}^{\ell_{k-1}}$ the successive subwords for every $1 \leqslant i \leqslant N'_k$. Formally,

$$\tilde{w}_{h,i} := \tilde{w}_h|_{(u_h^x + [\![1+(i-1)\ell_{k-1}, i\ell_{k-1}]\!])} \text{ and } \sigma^{u_h^x + (i-1)\ell_{k-1}}(\tilde{w}_{h,i}) = b_{k-1}.$$

Consider now a fixed position h. We will show that J_h^* is equal to a disjoint union of N'_k vertical strips $(J_{h,i}^*)_{i=1}^{N'_k}$ of the following forms:

- the initial strip $J^*_{j,1}$,

$$u_h + (\llbracket 1+\ell_{k-2}, \ell_{k-1}\rrbracket \times \llbracket c_{h,1}, d_{h,1}\rrbracket) \subseteq J^*_{j,1} \subseteq (u_h + \llbracket 1, \ell_{k-1}\rrbracket) \times \llbracket c_{h,1}, d_{h,1}\rrbracket;$$

- the intermediate strips, $J^*_{h,i}$, $1 < i < N'_k$,

$$J^*_{h,i} = u_h + (\llbracket (i-1)\ell_{k-1}+1, i\ell_{k-1}\rrbracket \times \llbracket c_{h,i}, d_{h,i}\rrbracket),$$

- the terminal strip J^*_{h,N'_k},

$$u_h + \left(\llbracket 1+(N'_k-1)\ell_{k-1}, \ell_k - \ell_{k-2}\rrbracket \times \llbracket c_{h,N'_k}, d_{h,N'_k}\rrbracket\right) \subseteq$$
$$\subseteq J^*_{h,N'_k} \subseteq u_h + \left(\llbracket 1+(N'_k-1)\ell_{k-1}, \ell'_k\rrbracket \times \llbracket c_{h,N'_k}, d_{h,N'_k}\rrbracket\right).$$

Here for each $i \in \llbracket 1, N'_k\rrbracket$, the values $1 \leqslant c_{h,i}, d_{h,i} \leqslant \ell'_k$ are integers that represent the vertical length of each strip. Note that it possible that $c_{h,i} > d_{h,i}$, which denotes an empty strip $J^*_{h,i}$.

Indeed, for a fixed $1 \leqslant i \leqslant N'_k$, we first consider the previous J_g, $1 \leqslant g < h$, that intersects the strip $J_{h,i}$ so that the word $\widetilde{w}_g$ overlaps $\widetilde{w}_h$ on a power of b_{k-1} (see item (2) of Lemma 3.8). Then $c_{h,i}$ is the largest upper level of those $J_g \cap J_{h,i}$, more precisely,

$$c_{h,i} = \max_g \left\{u^y_g + \ell'_k + 1 : u^y_g \leqslant u^y_h, \left(u^x_h + (i-1)\ell_{k-1} + \llbracket 1, \ell_{k-1}\rrbracket\right) \subseteq \left(u^x_g + \llbracket 1, \ell'_k\rrbracket\right)\right\}, \quad (7)$$

and similarly $d_{h,i}$ is the smallest lower level of those $J_g \cap J_{h,i}$, formally we have

$$d_{h,i} = \min_g \left\{u^y_g + 1 : u^y_g \geqslant u^y_h, \left(u^x_h + (i-1)\ell_{k-1} + \llbracket 1, \ell_{k-1}\rrbracket\right) \subseteq \left(u^x_g + \llbracket 1, \ell'_k\rrbracket\right)\right\}. \quad (8)$$

We have just constructed the intermediate strips $J^*_{h,i}$ for $1 < i < N_k$, se Figure 2.

We now construct the initial strip (the terminal strip is constructed similarly). We intersect the remaining J_g with $J_{h,1}$. The terminal segment b^T_{k-1} of $\widetilde{w}_g$ overlaps the initial segment b^I_{k-1} of $\widetilde{w}_h$. By item (1) of Lemma 3.8, as $k-1$ is odd, b_{k-1} has the same structure as a_k, and hence the overlapping can only happen at their end segments of the form b_{k-2}. We have just proved that $J^*_{h,1}$ contains a small strip $\left(u_h + \llbracket 1+\ell_{k-2}, \ell_{k-1}\rrbracket\right) \times \llbracket c_{h,1}, d_{h,1}\rrbracket$ of base $b^I_{k-1}\backslash b_{k-2}$ and is included in a larger strip $\left(u_h + \llbracket 1, \ell_{k-1}\rrbracket\right) \times \llbracket c_{h,1}, d_{h,1}\rrbracket$ of base b_{k-1}. For the initial and terminal strip the vertical extension ($\llbracket c_{h,1}, d_{h,1}\rrbracket$ and $\llbracket c_{h,N'_k}, d_{h,N'_k}\rrbracket$) of the elements $J^*_{h,1}$ and J^*_{h,N'_k} are defined as in (7) and (8).

Let $K^*_{h,i} := \{v \in J^*_{h,i} : p_v = 0\}$. We show that

$$\mathrm{card}(K^*_{h,i}) \leqslant \left(1 - N^{-1}_{k-1}\right)^{-1} \mathrm{card}(J^*_{h,i}) f^B_k \text{ for every } 1 \leqslant i \leqslant N_k. \quad (9)$$

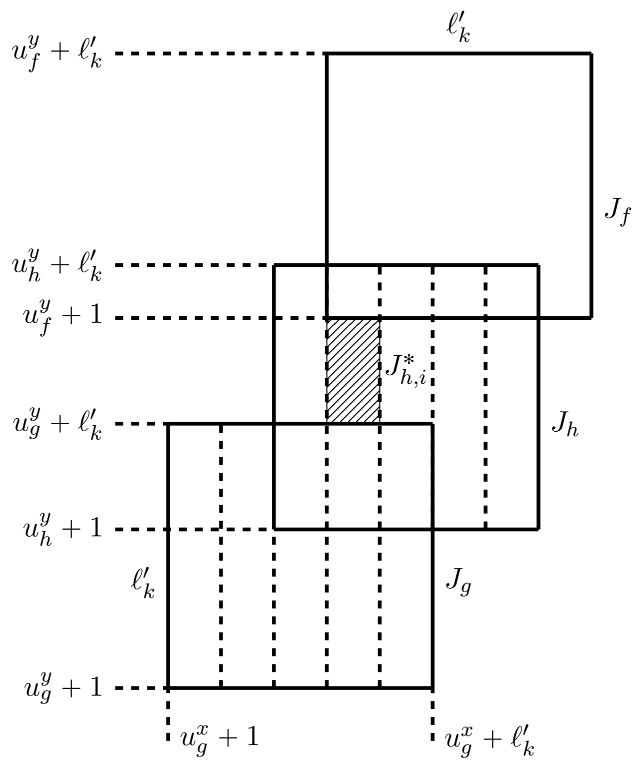

Figure 2: We represent the intermediate strip $J^*_{j,i}$ in J_h that is obtained after discarding J_g and J_f constructed before.

For the intermediate strips $J^*_{h,i}$, where $1 < i < N'_k$, we use the fact that $J^*_{h,i}$ is a square strip of base b_{k-1}, and the fact that the frequency f^B_{k-1} of the symbol 0 in the word b_{k-1} is identical to the frequency f^B_k of the symbol 0 in b_k. We have,

$$\text{card}(K^*_{h,i}) = \ell_{k-1}(d_{h,i} - c_{h,i} + 1) f^B_k = \text{card}(J^*_{h,i}) f^B_k.$$

For the initial strip $J^*_{h,1}$, we use the fact $J^*_{h,1}$ resembles largely a square strip of base b_{k-1}. We have,

$$\begin{aligned}
\text{card}(\ K^*_{h,i}) &\leqslant \ell_{k-1}(d_{h,1} - c_{h,1} + 1) f^B_k \\
&\leqslant \frac{\ell_{k-1}}{\ell_{k-1} - \ell_{k-2}} (\ell_{k-1} - \ell_{k-2})(d_{h,1} - c_{h,1} + 1) f^B_k \\
&\leqslant \big(1 - N^{-1}_{k-1}\big)^{-1} \text{card}(J^*_{h,i}) f^B_k.
\end{aligned}$$

We have proven (9) and by summing over $i \in [\![1, N'_k]\!]$ we have proven (6).

Item 2. As before we will consider I^A, but only consider the translates $u \in [\![0, n-\ell'_k]\!]^2$ such that $\tilde{\tilde{\pi}}(\sigma^u(p)|_{[\![1,\ell'_k]\!]^2}) \in \{a'_k, a''_k\}$. If $J_g \cap J_h \neq \varnothing$, the two projected words $\widetilde{w}_g = \tilde{\tilde{\pi}}(\sigma^{u_g}(p)|_{[\![1,\ell'_k]\!]^2})$ and $\widetilde{w}_h = \tilde{\tilde{\pi}}(\sigma^{u_h}(p)|_{[\![1,\ell'_k]\!]^2})$ may coincide in three ways: either $u_g^x = u_h^x$ and $\widetilde{w}_g = \widetilde{w}_h$, or $\widetilde{w}_g$ and $\widetilde{w}_h$ intersect on their markers, or $\widetilde{w}_g$ and $\widetilde{w}_h$ intersect on their initial and terminal segments, as in Lemma 3.9.

We redefine again I^A by clustering into a unique rectangle formed by adjacent squares where the overlap occurs in the whole word, that is, we group the squares J_g and J_h that pairwise satisfy $J_g \cap J_h \neq \varnothing$, $u_g^x = u_h^x$, $\widetilde{w}_g = \widetilde{w}_h$ and $|u_g^y - u_h^y| < \ell'_k$. Then, after re-indexing I^A, one obtains,

$$J^A = \bigcup_{h=1}^{H} J_h, \quad J_h = u_h + \left([\![1, \ell'_k]\!] \times [\![1, d_h]\!]\right),$$

where d_h is the final height of each rectangle obtained after the clustering. Thus $w_h^* = \sigma^{u_h}(p)|_{[\![1,\ell'_k]\!] \times [\![1,d_h]\!]}$ is a vertically aligned pattern whose projection $\widetilde{w}_h = \tilde{\tilde{\pi}}(w_h^*)$ is a word of the form a'_k or a''_k, and such that whenever $J_g \cap J_h \neq \varnothing$, $\widetilde{w}_g$ and $\widetilde{w}_h$ intersect at their initial and terminal segments, see Figure 3.

We now show that an index $v = (v^x, v^y) \in J^A$ may belong to at most two rectangles J_f and J_h. Indeed, by construction, as $u_g^x \neq u_h^x$, if v^x belongs to two overlapping words of the form a'_k, a''_k, then v^x belongs to either the intersection of the two markers $1^{(N'_k-1)\ell_{k-1}}$ or the intersection of the terminal segment a_{k-1}^T of a''_k and the initial segment a_{k-1}^I of a'_k. In both cases described in Lemma 3.9 we exclude the overlapping of a third word of the form a'_k, a''_k, thus we exclude the fact that v may belong to a third rectangle J_g with $u_g^x \neq u_f^x$ and $u_g^x \neq u_h^x$. Then

$$\begin{aligned}
\mathrm{card}(K^A) &= \sum_{v \in J^A} \mathbb{1}_{(p(v)=0)} \\
&\leqslant \sum_{h=1}^{H} \sum_{v \in (u_h + [\![1,\ell'_k]\!] \times [\![1,d_h]\!])} \mathbb{1}_{(p(v)=0)} \leqslant \sum_{h=1}^{H} f_{k-1}^A \ell_{k-1} d_h \\
&\leqslant \frac{f_{k-1}^A \ell_{k-1}}{\ell'_k} \sum_{h=1}^{H} \sum_{v \in J^A} \mathbb{1}_{v \in (u_h + [\![1,\ell'_k]\!] \times [\![1,d_h]\!])} = \frac{f_{k-1}^A}{N'_k} \sum_{v \in J^A} \sum_{h=1}^{H} \mathbb{1}_{(v \in J_h)} \\
&\leqslant \frac{2 f_{k-1}^A}{N'_k} \mathrm{card}(J^A).
\end{aligned}$$

Which concludes our proof. □

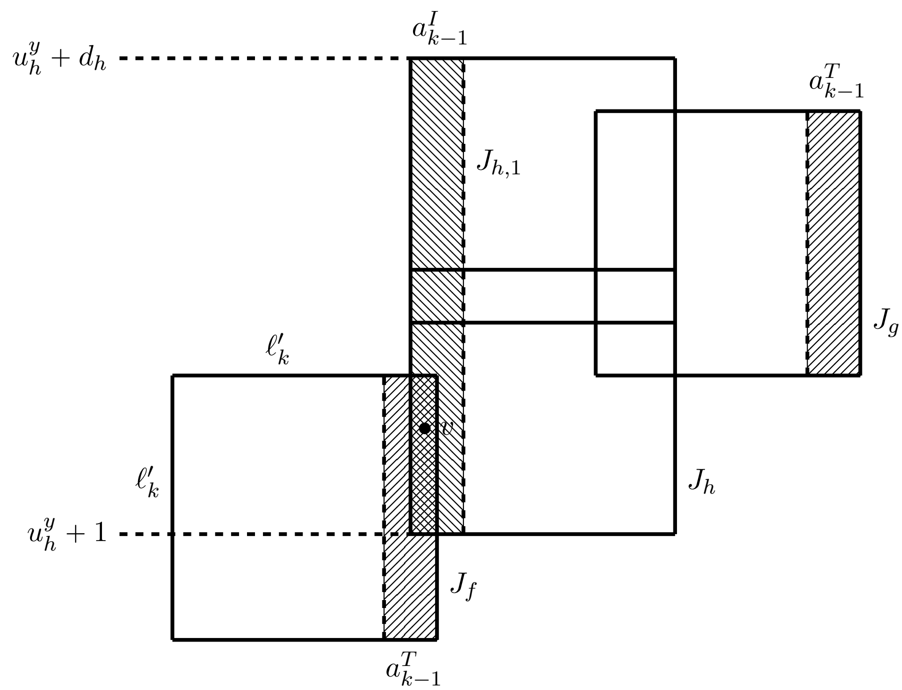

Figure 3: We represent a clustering of two squares of the kind J_h that intersects on the right J_g along their markers and on the left J_f along their initial and terminal segments.

4 Analysis of the zero-temperature limit

Proof of Lemma 2.22. Item 1. Let μ be a measure satisfying $\text{supp}(\mu) \subseteq X_k = \langle L_k \rangle$ which is ergodic. Recall that Birkhoff's ergodic theorem extends to actions of countable amenable groups as long as the average is taken over a tempered Følner sequence [32]. As the sequence $(\Lambda_n)_{n\in\mathbb{N}}$ with $\Lambda_n = [\![-n,n]\!]^d$ is tempered in $\mathbb{Z}^d$, it follows that for μ-almost every point x

$$\mu([\mathscr{F}]) = \lim_{n\to+\infty} \frac{\text{card}(\{u \in \Lambda_n : \sigma^u(x) \in [\mathscr{F}]\})}{\text{card}(\Lambda_n)}.$$

We choose such a point $x \in \langle L_k \rangle$ and $s \in [\![1,\ell_k]\!]^2$ such that $y = \sigma^s(x)$ and all its translates $\sigma^{t\ell_k}(y)$, $t \in \mathbb{Z}^2$, satisfy $\sigma^{t\ell_k}(y|_{[\![1,\ell_k]\!]^2}) \in L_k$. By taking a sub-sequence multiple of ℓ_k and by taking a set $\widetilde{\Lambda}_n$ tiled by translates of the square $[\![1,\ell_k]\!]^2$,

$$\widetilde{\Lambda}_n := [\![-n\ell_k, n\ell_k - 1]\!]^2,$$

one obtains

$$\mu([\mathscr{F}]) = \lim_{n\to+\infty} \frac{\mathrm{card}(\{u \in \widetilde{\Lambda}_n - s : \sigma^u(y) \in [\mathscr{F}]\})}{\mathrm{card}(\widetilde{\Lambda}_n)},$$
$$= \lim_{n\to+\infty} \frac{\mathrm{card}(\{u \in \widetilde{\Lambda}_n : \sigma^u(y) \in [\mathscr{F}]\})}{\mathrm{card}(\widetilde{\Lambda}_n)}.$$

By definition of L_k, every pattern in L_k is globally admissible and thus locally admissible,

$$\forall\, t \in [\![-n, n-1]\!]^2,\ \forall\, v \in [\![0, \ell_k - D]\!]^2,\ \sigma^{v+t\ell_k}(y)|_{[\![1,D]\!]^2} \notin [\mathscr{F}].$$

As $\mathrm{card}([\![0, \ell_k - 1]\!]^2 \backslash [\![0, \ell_k - D]\!]^2) \leqslant 2D\ell_k$, we have

$$\mathrm{card}(\{u \in \widetilde{\Lambda}_n : \sigma^u(y) \in [\mathscr{F}]\}) \leqslant (2n)^2 2D\ell_k,$$
$$\mathrm{card}(\widetilde{\Lambda}_n) = (2n)^2\ell_k^2.$$

Therefore we get that $\mu([\mathscr{F}]) \leqslant 2D/\ell_k$.

Item 2. Let $\widetilde{w} \in \widetilde{B}_k$ be the word whose density of zeroes realizes the maximum value f_k^B. By Lemma 2.18, $\widetilde{w}$ is a subword of some $\widetilde{x} \in \widetilde{X}$. Let $\widetilde{\widetilde{x}}$ be the vertically aligned configuration corresponding to $\widetilde{x}$. By the simulation theorem, $\widetilde{\widetilde{x}} = \widehat{\Pi}(\widehat{x})$ for some $\widehat{x} \in \widehat{X}$. Let $\widehat{w} := \widehat{x}|_{[\![1,\ell_k]\!]^2}$. By duplicating the symbol 0 we obtain,

$$\mathrm{card}(B_k) \geqslant \mathrm{card}\big(\{w \in \mathscr{A}^{[\![1,\ell_k]\!]^2} : \Gamma(w) = \widehat{w}\}\big) = 2^{\ell_k^2 f_k^B(\widetilde{w})},$$
$$h_{top}(X_k^B) = \frac{1}{\ell_k^2}\ln(\mathrm{card}(B_k)) \geqslant \ln(2) f_k^B.$$

where Γ has been defined in Equation 3.

Item 3. let μ_k^B be an ergodic measure of maximal entropy of X_k^B. Then

$$\mathrm{supp}(\mu_k^B) \subseteq X_k^B \quad \text{and} \quad P(\beta_k\varphi) \geqslant h(\mu_k^B) - \beta_k\mu_k^B([\mathscr{F}]) \geqslant \ln(2) f_k^B - 2D\frac{\beta_k}{\ell_k}.$$

Which is what we wanted to prove. □

Proof of Lemma 2.24. As the pressure of $\beta_k\varphi$ is non-negative (the two configurations 1^∞ and 2^∞ belong to X and φ is identically zero on X), we have

$$h(\mu_{\beta_k}) - \int \beta_k\varphi\, d\mu_{\beta_k} = P(\beta_k\varphi) \geqslant 0, \quad \mu_{\beta_k}([\mathscr{F}]) \leqslant \frac{h(\mu_{\beta_k})}{\beta_k} \leqslant \frac{\ln(\mathrm{card}(\mathscr{A}))}{\beta_k},$$
$$\Sigma^2(\mathscr{A}) \backslash [M_k'] \subseteq \bigcup_{u \in [\![0, R_k' - D]\!]^2} \sigma^{-u}([\mathscr{F}]), \quad \mu_{\beta_k}(\Sigma^2(\mathscr{A}) \backslash [M_k']) \leqslant \frac{{R_k'}^2}{\beta_k}\ln(\mathrm{card}(\mathscr{A})).$$

□

Proof of Lemma 2.26. By definition of relative entropy

$$\begin{aligned}
H\Big(\mathscr{P}^{[\![1,n]\!]^2}, \mu_{\beta_k}\Big) &= H\Big(\mathscr{P}^{[\![1,n]\!]^2} \bigvee \widetilde{\mathscr{G}}^{[\![1,n]\!]^2} \bigvee \mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) \\
&= H\Big(\mathscr{P}^{[\![1,n]\!]^2} \mid \widetilde{\mathscr{G}}^{[\![1,n]\!]^2} \bigvee \mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) \\
&\quad + H\Big(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2} \mid \mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) + H\Big(\mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big).
\end{aligned}$$

The first term of the right hand side is the relative entropy at scale ℓ'_k that requires a special treatment. The third term is computed using the estimate in lemma 2.24 (the function that maps $\varepsilon \in (0, e^{-1})$ to $H(\varepsilon)$ is increasing),

$$\begin{aligned}
H\big(\mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\big) &= \sum_{P \in \mathscr{U}_k^{[\![0,n-R'_k]\!]^2}} -\mu_{\beta_k}(P) \ln(\mu_{\beta_k}(P)) \\
&\leqslant n^2 H(\mathscr{U}_k, \mu_{\beta_k}) \leqslant n^2 H(\varepsilon_k).
\end{aligned}$$

We now compute the term in the middle. We choose $\varepsilon'_k > \varepsilon_k$ and define

$$U_n := \Big\{x \in \Sigma^2(\mathscr{A}) : \mathrm{card}\,\big\{u \in [\![0, n-R'_k]\!]^2 : \sigma^u(x) \in [M'_k]\big\} \geqslant n^2(1-\varepsilon'_k)\Big\}.$$

By the $\mathbb{Z}^d$-version of Birkhoff's ergodic theorem we have that

$$\lim_{n \to +\infty} \mu_{\beta_k}(U_n) = 1.$$

Let $(\mu_x)_{x \in \Sigma}$ be the family of conditional measures with respect to $\mathscr{U}_k^{[\![0,n-R'_k]\!]}$. We have

$$\begin{aligned}
H\left(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2} \mid \mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\right) &= \int H\Big(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2}, \mu_x\Big) d\mu_{\beta_k}(x) \\
&= \int_{U_n} H\Big(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2}, \mu_x\Big) d\mu_{\beta_k}(x) + \\
&\quad + \int_{\Sigma^2(\mathscr{A}) \backslash U_n} H\Big(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2}, \mu_x\Big) d\mu_{\beta_k}(x) \\
&\leqslant \int_{U_n} H\Big(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2}, \mu_x\Big) d\mu_{\beta_k}(x) + \\
&\quad + n^2 \mu_{\beta_k}\left(\Sigma^2(\mathscr{A}) \backslash U_n\right) \ln(\mathrm{card}(\widetilde{\mathscr{A}})),
\end{aligned}$$

and therefore

$$\limsup_{n \to +\infty} \frac{1}{n^2} H\Big(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2} \mid \mathscr{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) \leqslant \limsup_{n \to +\infty} \int_{U_n} \frac{1}{n^2} H\Big(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2}, \mu_x\Big) d\mu_{\beta_k}(x).$$

We now consider a fixed $x \in U_n$. We compute the cardinality of elements in $\widetilde{\mathscr{G}}^{[\![1,n]\!]^2}$ that are compatible with the constraint

$$\mathrm{card}\{u \in [\![0, n - R'_k]\!]^2 : \sigma^u(x) \in [M'_k]\} \geqslant n^2(1 - \varepsilon'_k).$$

Let I be the subset

$$I := \left\{u \in [\![0, n - R'_k]\!]^2 : \sigma^u(x) \in [M'_k]\right\}.$$

Since $x \in U_n$, we have

$$\frac{\mathrm{card}(I)}{n^2} \geqslant 1 - \varepsilon'_k.$$

Let $J \subseteq I$ be a maximal subset satisfying for every $u, v \in J$,

$$\|u - v\|_\infty \geqslant \frac{1}{2} R'_k.$$

For every $u \in J$, consider

$$I_u := \Big\{v \in I : \|u - v\|_\infty < \frac{1}{2} R'_k\Big\}.$$

By maximality of J we have $I = \bigcup_{u \in J} I_u$. We first observe that the sets $\Big(u + [\![1, \lceil R'_k/2 \rceil]\!]^2\Big)_{u \in J}$ are pairwise disjoint. Then

$$\mathrm{card}(J) \leqslant \frac{4n^2}{R'^2_k}.$$

We also observe that for every $v_1, v_2 \in I_u$, $\|v_1 - v_2\|_\infty < R'_k$ and

$$\Big(v_1 + [\![1, R'_k]\!]^2\Big) \bigcap \Big(v_2 + [\![1, R'_k]\!]^2\Big) \neq \varnothing.$$

For every $u \in I$ let

$$K_u := \bigcup_{v \in I_u} \left(v + [\![1, R'_k]\!]^2\right) \subset [\![1, n]\!]^2.$$

For every $v \in I_u$, we have

$$x|_{v + [\![1, R'_k]\!]^2} \in [M'_k].$$

In particular the pattern $x|_{v + [\![1, R'_k]\!]^2}$ is locally $\mathscr{F}$-admissible and satisfies the constrain that all the $\widetilde{\mathscr{A}}$-symbols are vertically aligned in $v + [\![1, R'_k]\!]^2$. Using that K_u is connected as a Cayley subgraph of $\mathbb{Z}^2$ with the canonical generators, one obtains that the $\widetilde{\mathscr{A}}$-symbols are also vertically aligned in K_u.

The width of K_u is less than $2R'_k$, so the cardinality of possible patterns $p \in \widetilde{\mathscr{A}}^{K_u}$ satisfying the constrain that the $\widetilde{\mathscr{A}}$-symbols are vertically aligned is bounded by $\mathrm{card}(\widetilde{\mathscr{A}})^{2R'_k}$. The cardinality of set of patterns with support $\bigcup_{u \in J} K_u$ is thus bounded by

$$\left(\mathrm{card}(\widetilde{\mathscr{A}})^{2R'_k}\right)^{4n^2/R'^2_k} = \exp\left(\left[2R'_k \cdot \frac{4n^2}{R'^2_k}\right] \ln(\mathrm{card}(\widetilde{\mathscr{A}}))\right) = \exp\left(\frac{8n^2}{R'_k} \ln(\mathrm{card}(\widetilde{\mathscr{A}}))\right).$$

Since $\bigcup_{u \in J} K_u$ covers I, the cardinality of the set of patterns with support $[\![1,n]\!]^2 \setminus \bigcup_{u \in J} K_u$ is bounded by $\mathrm{card}(\widetilde{\mathscr{A}})^{n^2 \varepsilon'_k}$. We have proven that, for every $x \in U_n$,

$$H\left(\widetilde{\mathscr{G}}^{[\![1,n]\!]^2}, \mu_x\right) \leqslant \left(\frac{8n^2}{R'_k} + n^2 \varepsilon'_k\right) \ln(\mathrm{card}(\widetilde{\mathscr{A}})).$$

We conclude by letting $n \to +\infty$ and $\varepsilon'_k \to \varepsilon_k$. □

The proof of Lemma 2.27 requires the following intermediate result.

Lemma 4.1. Let n, ℓ be integers which satisfy $n > 2\ell > 2$, $\varepsilon \in (0,1)$, and let $S \subseteq [\![0, n-2\ell]\!]^2$ be a subset satisfying $\mathrm{card}(S) \geqslant n^2(1-\varepsilon)$. Let $\widehat{E}$ be the set

$$\widehat{E} := \left\{ w \in \widehat{\mathscr{A}}^{[\![1,n]\!]^2} : \forall\, u \in S,\ \sigma^u(w)|_{[\![1,2\ell]\!]^2} \in \mathscr{L}(\widehat{X}, 2\ell) \right\}.$$

Then

$$\frac{1}{n^2} \ln(\mathrm{card}(\widehat{E})) \leqslant \frac{1}{\ell} \ln(\mathrm{card}(\widetilde{\mathscr{A}})) + \frac{1}{\ell^2} \ln(C^{\widehat{X}}(\ell)) + \varepsilon \ln(\mathrm{card}(\widehat{\mathscr{A}})).$$

Proof. To simplify the notations we assume that n is a multiple of ℓ. We decompose the square $[\![1,n]\!]^2$ into a disjoint union of squares of size ℓ,

$$[\![1,n]\!]^2 = \bigcup_{v \in [\![0, \frac{n}{\ell}-1]\!]^2} \left(\ell v + [\![1,\ell]\!]^2\right).$$

We define the set of indexes v that intersect S, more precisely, we have

$$V := \left\{ v \in \left[\!\!\left[0, \frac{n}{\ell} - 2\right]\!\!\right]^2 : \left(\ell v + [\![0, \ell-1]\!]^2\right) \bigcap S \neq \varnothing \right\}.$$

Then for every $w \in \widehat{E}$, $v \in V$, and $u \in \left(\ell v + [\![0,\ell-1]\!]^2\right) \bigcap S$, therefore

$$\left(\ell v + [\![1+\ell, 2\ell]\!]^2\right) \subseteq \left(u + [\![1, 2\ell]\!]^2\right).$$

Since we are taking $u \in S$ we have that

$$\sigma^u(w)|_{[\![1,2\ell]\!]^2} \in \mathscr{L}(\widehat{X}, 2\ell),$$

and then

$$\sigma^{\ell v+(\ell,\ell)}(w)|_{[\![1,\ell]\!]^2} \in \mathscr{L}(\widehat{X},\ell).$$

The restriction of w on every square $\big(\ell v + [\![1+\ell, 2\ell]\!]^2\big)$ is globally admissible with respect to $\widehat{\mathscr{F}}$. Note that these squares are pairwise disjoint and the cardinality of their union is at least $n^2(1-\varepsilon)$, since

$$\mathrm{card}\left(\bigcup_{v\in V}\big(\ell v + [\![1+\ell,2\ell]\!]^2\big)\right) = \mathrm{card}\left(\bigcup_{v\in V}\big(\ell v + [\![0,\ell-1]\!]^2\big)\right) \geqslant \mathrm{card}(S).$$

Hence we proved that $\widehat{E}$ is a subset of the set of patterns w made of independent and disjoint words $(w_v)_{v\in V}$, with $w_v \in \mathscr{L}(\widehat{X},\ell)$, and of arbitrary symbols on $[\![0,n-2\ell]\!]^2\backslash S$ of size at most εn^2. Using the trivial bound $\mathrm{card}(\mathscr{L}(\widetilde{X},\ell)) \leqslant \mathrm{card}(\widetilde{\mathscr{A}})^\ell$, we have

$$\mathrm{card}(\widehat{E}) \leqslant \Big(\mathrm{card}(\widetilde{\mathscr{A}})^\ell \cdot C^{\hat{X}}(\ell)\Big)^{(n/\ell)^2} \cdot \mathrm{card}(\widehat{\mathscr{A}})^{\varepsilon n^2}$$

and therefore

$$\frac{1}{n^2}\ln(\mathrm{card}(\widehat{E})) \leqslant \frac{1}{\ell}\ln(\mathrm{card}(\widetilde{\mathscr{A}})) + \frac{1}{\ell^2}\ln(C^{\hat{X}}(\ell)) + \varepsilon\ln(\mathrm{card}(\widehat{\mathscr{A}})).$$

□

Proof of Lemma 2.27.

Proof of item 1. Using the $\mathbb{Z}^d$-version of Birkhoff's ergodic theorem and Lemma 2.24, it follows that for almost every $x \in \Sigma^2(\mathscr{A})$,

$$\lim_{n\to+\infty}\frac{1}{n^2}\mathrm{card}\big(\{u \in [\![0,n-R'_k]\!]^2 : \sigma^u(x) \in [M'_k]\}\big) = \mu_{\beta_k}([M'_k])$$

and

$$\lim_{n\to+\infty}\frac{1}{n^2}\mathrm{card}\big(\{u \in [\![1,n]\!]^2 : \pi(x(u)) = \tilde{a}\}\big) = \mu_{\beta_k}(\widetilde{G}_{\tilde{a}}), \quad \forall\, \tilde{a} \in \widetilde{\mathscr{A}}.$$

We choose $n > R'_k$. An element of the partition $\widetilde{\mathscr{G}}^{[\![1,n]\!]^2} \bigvee \mathscr{U}^{[\![0,n-R'_k[\![^2}$ is of the form $\widetilde{G}_p \cap U_S$ where $p \in \widetilde{\mathscr{A}}^{[\![1,n]\!]^2}$ is a pattern and $S \subseteq [\![0,n-R'_k]\!]^2$ is a subset, where

$$U_S := \big\{x \in \Sigma^2(\mathscr{A}) : \forall\, u \in S,\ \sigma^u(x) \in [M'_k],\ \forall\, u \in [\![0,n-R'_k]\!]^2\backslash S,\ \sigma^u(x) \notin [M'_k]\big\},$$
$$\widetilde{G}_p := \{x \in \Sigma^2(\mathscr{A}) : \Pi(x|_{[\![1,n]\!]^2}) = p\}.$$

Let $\eta < \mu_{\beta_k}(\widetilde{G}_0)$. Lemma 2.24 implies $\mu_{\beta_k}\big(\Sigma^2(\mathscr{A})\backslash[M'_k]\big) \leqslant \varepsilon_k$. Using again Birkhoff's theorem we get

$$\lim_{n\to+\infty}\frac{1}{n^2}\mathrm{card}\left(\{u \in [\![0,n-R'_k]\!] : \sigma^u(x) \in [M'_k]\}\right) > 1-\varepsilon_k, \text{for } \mu_{\beta_k}\text{-a.e. } x,$$

$$\lim_{n \to +\infty} \mu_{\beta_k} \Big(\bigcup_{S \subseteq [\![0, n-R'_k]\!]^2} \{U_S : \mathrm{card}(S) > n^2(1-\varepsilon_k)\} \Big) = 1. \tag{10}$$

For n large enough, we choose $S \subseteq [\![0, n-R'_k]\!]^2$ such that $U_S \neq \varnothing$ and $\mathrm{card}(S) \geqslant n^2(1-\varepsilon_k)$. By definition of M'_k and T'_k (see page 16), if $x \in U_S$, then for every $u \in S$, $\sigma^u(x)|_{[\![1,R'_k]\!]^2}$ is a locally admissible pattern with respect to $\mathscr{F}$ and

$$\sigma^{u+T'_k}(x)|_{[\![1,2\ell'_k]\!]^2} \in \mathscr{L}(X, 2\ell'_k).$$

Define for every $n > R'_k$ and every pattern $p \in \widetilde{\mathscr{A}}^{[\![1,n]\!]^2}$ the set

$$K_n(p) := \{u \in [\![1, n]\!]^2 : p(u) = 0\}.$$

As $\mu_{\beta_k}(\widetilde{G}_0) > \eta$, it follows by Birkhoff's ergodic theorem

$$\lim_{n \to +\infty} \mu_{\beta_k} \left(\bigcup_p \left\{ \widetilde{G}_p : \mathrm{card}(K_n(p)) > n^2\eta \right\} \right) = 1. \tag{11}$$

From Equations 10 and 11, for large n, one can choose S and p such that $U_S \cap \widetilde{G}_p \neq \varnothing$, $\mathrm{card}(K_p) > n^2\eta$ and $\mathrm{card}(S) \geqslant n^2(1-\varepsilon_k)$. Using the notations in Definition 3.11 and the conclusions of Lemma 3.12, one obtains

$$T'_k + S \subseteq I(p, \ell'_k) \quad \text{and} \quad \tau'_k + I(p, \ell'_k) \subseteq J^A(p, \ell'_k) \sqcup J^B(p, \ell'_k) =: J^A \sqcup J^B,$$

therefore by our choice of S we obtain

$$n^2(1-\varepsilon_k) \leqslant \mathrm{card}(S) = \mathrm{card}(\tau'_k + T'_k + S) \leqslant \mathrm{card}\big(J^A \sqcup J^B\big) \leqslant n^2. \tag{12}$$

Besides that, we have

$$n^2\eta \leqslant \mathrm{card}(K_n(p)) \leqslant \mathrm{card}(K^A \sqcup K^B) + n^2\varepsilon_k$$

and by the Lemma 3.13 we have

$$\mathrm{card}(K_n(p)) \leqslant \frac{2}{N'_k}\mathrm{card}(J^A) f^A_{k-1} + \big(1 - N^{-1}_{k-1}\big)^{-1} \mathrm{card}(J^B) f^B_{k-1} + n^2\varepsilon_k.$$

We divide each term by n^2 and take the limit with $n \to +\infty$, $\varepsilon \to \varepsilon_k$, and $\eta \to \mu_{\beta_k}([0])$.

Proof of item 2. We now assume that k is even and choose η such that $\mu_{\beta_k}(\widetilde{G}_1) > \eta$. We choose $p \in \widetilde{\mathscr{A}}^{[\![1,n]\!]^2}$ such that $\widetilde{G}_p \cap U_S \neq \varnothing$ and

$$\mathrm{card}\left(\left\{u \in [\![1, n]\!]^2 : p(u) = 1\right\}\right) > \eta n^2. \tag{13}$$

Let $x \in \widetilde{G}_p \cap U_S$ and $(\mu_x)_{x\in\Sigma}$ be the family of conditional measures with respect to the partition $\widehat{\mathscr{G}}^{[\![1,n]\!]^2} \bigvee \mathscr{U}^{[\![0,n-R_k[\![^2}$. We use the trivial upper bound of the entropy, so

$$H(\mathscr{G}^{[\![1,n]\!]^2}, \mu_x) \leqslant \ln(\mathrm{card}(E_{p,S})) \tag{14}$$

where

$$E_{p,S} := \left\{w \in \mathscr{A}^{[\![1,n]\!]^2} : \Pi(w) = p \ \text{ and } \ \forall u \in S,\ \sigma^{u+T'_k}(w)|_{[\![1,2\ell'_k]\!]^2} \in \mathscr{L}(X, 2\ell'_k)\right\}.$$

Also consider

$$\widehat{E}_{p,S} := \Gamma(E_{p,S}).$$

Note that every pattern in $E_{p,S}$ is obtained from a pattern in $\widehat{E}_{p,S}$ by duplicating the symbol 0. Using Lemma 4.1 we conclude that

$$\ln(\mathrm{card}(E_{p,S})) \leqslant \ln(\mathrm{card}(\widehat{E}_{p,S})) + \mathrm{card}(K_n(p))\ln(2),$$
$$\frac{1}{n^2}\ln(\mathrm{card}(\widehat{E}_{p,S})) \leqslant \frac{1}{\ell'_k}\ln(\mathrm{card}(\widetilde{\mathscr{A}})) + \frac{1}{{\ell'_k}^2}\ln(C'_k) + \varepsilon_k \ln(\mathrm{card}(\widehat{\mathscr{A}})),$$

thus

$$\begin{aligned}\frac{1}{n^2}\ln(\mathrm{card}(E_{p,S})) \ &\leqslant \ \left(\frac{2}{N'_k}\mathrm{card}(J^A)f^A_{k-1} + (1-N^{-1}_{k-1})^{-1}\mathrm{card}(J^B)f^B_{k-1} + n^2\varepsilon_k\right)\frac{\ln(2)}{n^2} + \\ &\quad + \frac{1}{\ell'_k}\ln(\mathrm{card}(\widetilde{\mathscr{A}})) + \frac{1}{{\ell'_k}^2}\ln(C'_k) + \varepsilon_k\ln(\mathrm{card}(\widehat{\mathscr{A}})).\end{aligned} \tag{15}$$

The symbol 1 does not appear in $J^B = J^B(p, \ell'_k)$, thus

$$\left\{u \in [\![1,n]\!]^2 : p(u) = 1\right\} \subset J^A \sqcup \left([\![1,n]\!]^2 \backslash (J^A \sqcup J^B)\right).$$

Since we are assuming (13), using (12) and the fact that $J^A \cap J^B = \varnothing$, we obtain

$$\mathrm{card}\left([\![1,n]\!]^2 \backslash (J^A \sqcup J^B)\right) \leqslant n^2\varepsilon_k$$

$$\mathrm{card}(J^A) \geqslant n^2\Big(\eta - \varepsilon_k\Big) \quad \text{and} \quad \mathrm{card}(J^B) \leqslant n^2\Big(1 - \eta + \varepsilon_k\Big). \tag{16}$$

By replacing the upper bound for $\mathrm{card}(J^B)$ given in (16) and $\mathrm{card}(J^A) \leqslant n^2$ in (15) we obtain that

$$\begin{aligned}\frac{1}{n^2}\ln(\mathrm{card}(E_{p,S})) \ &\leqslant \ \left(\frac{2}{N'_k}f^A_{k-1} + (1-N^{-1}_{k-1})^{-1}(1-\eta+\varepsilon_k)f^B_{k-1} + \varepsilon_k\right)\ln(2) + \\ &\quad + \frac{1}{\ell'_k}\ln(\mathrm{card}(\widetilde{\mathscr{A}})) + \frac{1}{{\ell'_k}^2}\ln(C'_k) + \varepsilon_k\ln(\mathrm{card}(\widehat{\mathscr{A}})).\end{aligned} \tag{17}$$

By integrating with respect to μ_{β_k} in both sides and taking the limit when $n \to +\infty$ we obtain item (2) of Lemma 2.27. Item (3) is analogous. □

5 Results on computability

In this last section we prove the two bounds on the relative complexity and reconstruction functions. The subshift of finite type $\widehat{X} = \Sigma^2(\widehat{\mathscr{A}}, \widehat{\mathscr{F}})$ in the Aubrun-Sablik construction [1] as described in Theorem 2.9 is composed of four layers, that is, it is a subshift of a product of four subshifts of finite type, which is itself described by a finite set of forbidden patterns which impose conditions on how the layers superpose. See Figure 14 of [1]. The layers are:

1. `Layer` 1: The set of all configurations $x \in \widetilde{\mathscr{A}}^{\mathbb{Z}^2}$ that are vertically aligned, that is, $x_u = x_{u+(0,1)}$ for every $u \in \mathbb{Z}^2$.
2. `Layer` 2: $\mathbf{T}_{\texttt{Grid}}$ A subshift of finite type extension of a sofic subshift which is generated by the substitution given in Figure 3 of [1]. The sofic subshift induces infinite vertical "strips" of computation which are of width 2^n for every $n \in \mathbb{N}$ and occur with bounded gaps (horizontally) in any configuration. It also encodes a "clock" on every computation strip of width 2^n, which counts and restarts periodically every $2^{2^n} + 2$ vertical steps in its strip.
3. `Layer` 3: $\mathscr{M}_{\texttt{Forbid}}$ A subshift of finite type given by Wang tiles which replicates, on top of each clock determined by the previous layer, the space-time diagram of a Turing machine which enumerates all forbidden patterns of Z. It also communicates information from the space-time diagram to the fourth layer.
4. `Layer` 4: $\mathscr{M}_{\texttt{Search}}$ A subshift of finite type given by Wang tiles which simulates a Turing machine which serves the purpose of checking whether the patterns enumerated by the third layer appear in the first layer. Each machine searches for forbidden patterns in a "responsibility zone" which is determined by the hierarchical structure of `Layer` 2.

The rules between the four layers described in [1] force the Turing machine space-time diagrams to occur in every strip, and to restart their computation after an exponential number of steps. This ensures that every configuration witnesses every step of computation in a relatively dense set, and that every forbidden pattern is written on the tape by the Turing machine in every large enough strip. The fourth layer uses the information from the third layer to search for occurrences of the forbidden patterns in the first layer and thus discards any configuration in the first layer where one of these patterns occurs.

In the proofs that follow, we shall use nomenclature of [1] and refer to explicit parts and bounds associated to their construction, so the reader should bear in mind that this section is not meant to be self-contained. However, we shall aim to explain our arguments in such a way that at least they can be understood intuitively without the need to refer to [1].

Proof of Proposition 2.14. Let us denote by $C_n(\texttt{Layer}_k(\widehat{X}))$ the complexity of the projection to the k-th layer. and by $C_n(\texttt{Layer}_k(\widehat{X})|\texttt{Layer}_j(\widehat{X}))$ the complexity of the projection to the k-th layer given that there is a fixed pattern on the j-th layer. Clearly we have that

$$C^{\widehat{X}}(n) \leqslant C_n(\texttt{Layer}_1(\widehat{X})) \cdot C_n(\texttt{Layer}_2(\widehat{X})) \cdot C_n(\texttt{Layer}_3(\widehat{X})|\texttt{Layer}_2(\widehat{X})) \cdot \\ \cdot C_n(\texttt{Layer}_4(\widehat{X})|\texttt{Layer}_2(\widehat{X})).$$

- $\texttt{Layer}$ 1: As this layer is given by all $x \in \widetilde{\mathscr{A}}^{\mathbb{Z}^2}$ so that $x_u = x_{u+(0,1)}$ for every $u \in \mathbb{Z}^2$, a trivial upper bound for the complexity is

  $$C_n(\texttt{Layer}_1(\widehat{X})) = \mathscr{O}(|\widetilde{\mathscr{A}}|^n).$$

  In fact, as in the end the only configurations which are allowed are those whose horizontal projection lies in the effective subshift $\mathbb{Z}$, a better bound is given by $C_n(\texttt{Layer}_1(\widehat{X})) = \mathscr{O}(\exp(n\, h_{\text{top}}(\widehat{X})))$. For simplicity, we shall just keep the trivial bound.

- $\texttt{Layer}$ 2: The complexity of every substitutive subshift in $\mathbb{Z}^2$ is $\mathscr{O}(n^2)$. To see this, suppose that the substitution sends symbols of some alphabet $\mathscr{A}_2$ to $n_1 \times n_2$ arrays of symbols. By definition, every pattern of size n occurs in a power of the substitution. If k is such that $\min\{n_1, n_2\}^{k-1} \leqslant n \leqslant \min\{n_1, n_2\}^k$, then necessarily any pattern of size n occurs in the concatenation of at most 4 k-powers of the substitution. There are $|\mathscr{A}_2|^4$ choices for the k-powers and at most $(\max\{n_1, n_2\}^k)^2 \leqslant (n \max\{n_1, n_2\})^2$ choices for the position of the pattern. It follows that there are at most $(|\mathscr{A}_2|^4 \max\{n_1, n_2\}^2)n^2 = \mathscr{O}(n^2)$ patterns of size n. We obtain,

  $$C_n(\texttt{Layer}_2(\widehat{X})) = \mathscr{O}(n^2)$$

  The reader can find further information on dynamical systems generated by subtitutions in [39, 19].

- $\texttt{Layer}$ 3: It can be checked directly from the Aubrun-Sablik construction that the symbols on the third layer satisfy the following property: if the symbols on the substitution layer are fixed, then for every $u \in \mathbb{Z}^2$ the symbol at position u is uniquely determined by the symbols at positions $u-(0,1), u-(1,1)$ and $u-(-1,1)$. In consequence, it follows that knowing the symbols at positions in the "U shaped region"

  $$U = (\{0\} \times [\![1, n-1]\!]) \cup ([\![0, n-1]\!] \times \{0\}) \cup (\{n-1\} \times [\![1, n-1]\!])$$

completely determines the pattern. Therefore, if this layer has an alphabet $\mathscr{A}_3$, we have

$$C_n(\texttt{Layer}_3(\widehat{X})|\texttt{Layer}_2(\widehat{X})) \leqslant |\mathscr{A}_3|^{3n-2} \leqslant \mathscr{O}(K_1^n),$$

for some positive integer K_1.

- $\texttt{Layer}$ 4: The same argument for Layer 3 holds for Layer 4. Therefore, if the alphabet of layer 4 is $\mathscr{A}_4$ we have that for some positive integer K_2,

$$C_n(\texttt{Layer}_4(\widehat{X})|\texttt{Layer}_2(\widehat{X})) \leqslant |\mathscr{A}_4|^{3n-2} \leqslant \mathscr{O}(K_2^n).$$

Putting the previous bounds together, we conclude that there is some constant $K > 0$ such that

$$C^{\widehat{X}}(n) = \mathscr{O}(n^2 K^n).$$

Which yields the desired bound on Proposition 2.14. □

We proved in Lemma 3.5 that $\widetilde{\mathscr{F}}$ satisfies the assumptions of Proposition 2.11. We now prove the upper bound of the reconstruction function $R^{\widehat{X}}\colon \mathbb{N} \to \mathbb{N}$ of $\widehat{X} = \Sigma^2(\widehat{\mathscr{A}}, \widehat{\mathscr{F}})$. Of course, a formal proof of these estimates would require a restatement of the construction of Aubrun-Sablik with all its details, which is out of the scope of this paper. Instead, we shall argue that the bounds we give suffice, making reference to the properties of the Aubrun-Sablik construction.

A description of $\widehat{\mathscr{F}}$ is given in [1] in an (almost) explicit manner for all layers except the substitution layer. For the substitution layer, a description of the forbidden patterns can be extracted in an explicit way from the article of Mozes [39].

The behavior of layers 2,3 and 4 in the Aubrun-Sablik construction is mostly independent of layer 1, except for the detection of forbidden patterns which leads to the forbidden halting state of the machine in the third layer. Because of that reason the analysis of the reconstruction function $R^{\widehat{X}}$ can be split into two parts:

1. **Structural:** Assuming that the contents of the first layer are globally admissible (the configuration in the first layer is an extension of a configuration from $\widetilde{X}$), we give a bound that ensures that the contents of layers $2, 3$ and 4 are globally admissible, that is:
   - The contents of layer 2 correspond to a globally admissible pattern in the substitutive subshift and the clock.
   - The contents of layer 3 and 4 correspond to valid space-time diagrams of Turing machines that correctly align with the clocks.
2. **Recursive:** A bound that ensures that the contents of the first layer are globally admissible. This bound will of course depend upon $R^{\widetilde{X}}$ and $T^{\widetilde{X}}$.

Finally, we are able to prove the upper bound for the reconstruction function given by Proposition 2.11.

Proof of Proposition 2.11. Let us begin with the structural part, as it is simpler and does not depend upon $\widetilde{X}$. Let p be a pattern with support B_n and assume that the first layer of p is thus globally admissible.

From Mozes's construction of SFT extensions for substitutions [39] it can be checked that any locally admissible pattern of support B_n of Mozes's SFT extension of a primitive substitution (The Aubrun-Sablik substitution is primitive) is automatically globally admissible. Let us take a support large enough such that the second layer of p occurs within four $4^k \times 2^k$ macrotiles of the substitution in any locally admissible pattern of that support.

Next, a clock runs on every strip of the Aubrun-Sablik construction. By the previous argument, the largest zone which intersects p in more than one position is of level at most k. Therefore its largest computation strip has horizontal length 2^k. In order to ensure that the clock starts on a correct configuration on every strip contained in p, we need to witness this pattern inside a locally admissible pattern which stacks $2^{2^k} + 2$ macrotiles of level k vertically. Therefore, the pattern p must occur inside four locally admissible patterns of length $4^k \times 2^k(2^{2^k} + 2)$. This ensures that the clocks in p are globally admissible.

Finally, if every clock occurring in p starts somewhere, then the contents of the third layer are automatically correct, as they are determined by clock every time it restarts. To check that the fourth layer is correct, we just need extend the horizontal length of our pattern twice, so that the responsibility zone of the largest strip is contained in it.

By the previous arguments, it would suffice to witness p inside a locally admissible pattern which contains in its center a 4×2 array of macrotiles of size $4^k \times 2^k(2^{2^k} + 2)$. As $4^{k-1} < N \leqslant 4^k$, there is a constant $C_0 > 0$ such that an estimate for this part of the reconstruction function can be written as

$$R^{\widehat{X}}_{\text{Struct}}(n) = \mathscr{O}(\sqrt{n} C_0^{\sqrt{n}}).$$

Let us now deal with the recursive part. We need to find a bound such that the word of length N occurring in the first layer of p is globally admissible. By definition of $R^{\widetilde{X}}$, it suffices to have p inside a pattern with support $B_{R^{\widetilde{X}}(N)}$ and check that the first layer is locally admissible with respect to $\widetilde{\mathscr{F}}$. In other words, we need to have the Turing machines check all forbidden words of length $R^{\widetilde{X}}(N)$ in this pattern. Luckily, the number of steps in order to do this is already computed in Aubrun and Sablik's article. After Fact 4.3 of [1] they show that, if $p_0, p_1, \ldots, p_r$ are the first $r+1$ patterns enumerated by $\mathscr{M}$, then the number of steps $S(p_0, \ldots, p_r)$ needed in a computation zone to completely check whether a pattern from $\{p_0, \ldots, p_r\}$ occurs in its responsibility zone of level m satisfies the bound,

$$S(p_0,\dots,p_r) \leqslant T(p_0,\dots,p_r) + (r+1)\max(|p_0|,\dots,|p_r|)m^2 2^{3m+1},$$

where $T(p_0,\dots,p_r)$ is the number of steps needed by $\mathscr{M}$ to enumerate the patterns $p_0, p_1, \dots, p_r$.

In our construction, we may rewrite their formula so that the number $S(R^{\widetilde{X}}(N))$ of steps needed to check that all forbidden patterns of length at most $R^{\widetilde{X}}(N)$ in a responsibility zone of level m satisfies the bound

$$\begin{aligned} S(R^{\widetilde{X}}(N)) &\leqslant \tau(R^{\widetilde{X}}(N)) + |\widetilde{\mathscr{A}}|^{R^{\widetilde{X}}(N)+1} R^{\widetilde{X}}(N) k^2 2^{3k+1} \\ &\leqslant P(n)|\widetilde{\mathscr{A}}|^N + |\widetilde{\mathscr{A}}|^{C_1 N+1} C_1 N m^2 2^{3m+1} \end{aligned}$$

Simplifying the above bound, it follows that there exists constants $C_2, C_3 > 0$ such that

$$S(R^{\widetilde{X}}(N)) \leqslant C_2 m^2 2^{3m+C_3 N}.$$

As N is constant, it follows that there is a smallest $\bar{m} = \bar{m}(N) \in \mathbb{N}$ such that $2^{\bar{m}} \geqslant C_4 N$ (so that the tape on the computation zone of level $\bar{m}$ can hold words of size $R_Z(N)$) and such that

$$C_2 \bar{m}^2 2^{3\bar{m}+C_3 N} \leqslant 2^{2^{\bar{m}}} + 2,$$

so that the number $2^{2^{\bar{m}}} + 2$ of computation steps in the zone of level $\bar{m}$ is enough to check all the words of size $R^{\widetilde{X}}(N)$. It follows that a bound for the recursive part of $R^{\widehat{X}}$ is given by

$$R^{\widehat{X}}_{\text{recursive}}(n) = \mathscr{O}(2^{\bar{m}+2^{\bar{m}(N)}}).$$

In order to turn this into an explicit asymptotic expression we need to find a suitable bound for $\bar{m}(N)$. Notice that if $m \geqslant 6$ we simultaneously have that $m^2 \leqslant 2^m$ and $4m \leqslant 2^{m-1}$. We may then write for $m \geqslant 6$,

$$C_2 m^2 2^{3m} e^{C_3 N} \leqslant C_2 2^{4m+C_3 N} \leqslant C_2 2^{C_3 N} 2^{2^{m-1}}.$$

Therefore, it suffices to find $\bar{m} = \bar{m}(N)$ such that

$$C_2 2^{C_3 N} \leqslant 2^{2^{\bar{m}-1}}.$$

From here, it follows that there is a constant $C_5 > 0$ such that any value of $\bar{m}$ satisfying

$$\bar{m} \geqslant C_5 + \log_2(N),$$

satisfies the above bound. We get that

$$R^{\widehat{X}}_{\text{recursive}}(n) = \mathscr{O}(N 2^{C_5 N}) = n 4^{C_5 n}.$$

Finally, putting together the structural and recursive asymptotic bounds, we obtain that there is a constant $K > 0$ such that

$$R^{\widehat{X}}(n) = \mathscr{O}(\max\{\sqrt{n}C_0^{\sqrt{n}}, \mathscr{O}(n4^{C_5 n})\}) = \mathscr{O}(nK^n).$$

Hence we get that

$$\limsup_{n\to+\infty} \log(R^{\widehat{X}}(n)) < +\infty.$$

Which is what we wanted to prove. □

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