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Tilings on different structures: exploration towards two problems

Author:
Sebastián Barbieri

Supervisor:
Dr. Nathalie Aubrun

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Abstract

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Tilings on different structures: exploration towards two problems

by Sebastián BARBIERI

We study the problem of tiling structures which are different from the usual group $\mathbb{Z}^d$. In the first part we show a class of finitely generated groups for which the G-subshift $X_{\leq 1}$, which consists on the functions from G to $\{0,1\}$ so that at most one $g \in G$ can map to 1, is not of sofic type. In the second part we study tilings over structures which are not groups and are generated by a special type of substitution. We define the emptiness problem and the possibility to simulate more complex substitutions in these structures and we show results in that direction for two specific examples. We end that section by constructing a partial order which under the assumption of a property preserves decidability of the emptiness problem monotonically.

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Contents

Introduction

Recent results in the theory of symbolic dynamics have found deep relations between dynamical properties of subshifts of finite type and computation theory. The first results in this direction were the studies by Wang where he asked about the decidability of the question if a set of tiles which can be assembled together by satisfying a finite number of local rules can tile the whole plane. This question became known as the domino problem and Wang proved that it was decidable if every Wang tiling[1] admitted a periodic point [Wan60]. The previous assumption about the existence of periodic points was called Wang's conjecture and it was proven to be false by Berger [Ber66] who constructed an aperiodic set of 20426 Wang tiles and also provided a proof of the undecidability of the domino problem. Subsequently, different authors have simplified Berger's proof yielding very elegant structures of $\mathbb{Z}^2$-SFT without periodic points, such as the Robinson tiling [Rob71] and more recently and in a different perspective the Kari-Culik tiling which contains only 13 tiles [Kar96, Cul96].

Other results exhibiting links between dynamical properties and computation have appeared lately. One of the most important in this sense is the result by Hochman and Meyerovitch [HM10] which characterizes the entropies of $\mathbb{Z}^d$-subshifts of finite type as the numbers which are right recursively enumerable. Another is a result which shows that every subshift defined by an effective set of forbidden patterns can be realized in the rows of a sofic $\mathbb{Z}^2$-subshift. First proven by Hochman [Hoc09] in $\mathbb{Z}^3$ and then generalized by Aubrun and Sablik [AS13] and Durand, Romashchenko and Shen [DRS10]. Doubtlessly this relationship between dynamical properties of subshifts defined over groups and computational theory is one of the reasons several researchers interested in symbolic dynamical systems are facing the problems from a computational perspective.

Even in the light of recent discoveries, still not much is known about the dynamics of $\mathbb{Z}^d$-subshifts, and even less is known for the more general case of G-subshifts when G is an arbitrary group. Even the problem of defining consistent dynamical invariants such as the entropy is not an trivial task in this general setting. There is a definition for entropy using Følner nets for amenable groups [CSC09] and a very recent extension for sofic groups [Bow10]. In spite of these difficulties, recent results suggest that for the case of general groups many properties could also be linked to computational related ideas. This rapport is dedicated to the study of this general case under the computational scope and to tilings of structures which are not groups but which can also be modeled as subshifts defined by a finite amount of information.

In the first part of this report we speak about an important class of groups such that the G-subshift which is defined over the alphabet $\mathcal{A} = \{0, 1\}$ and admits at most one occurrence of the symbol 1 in any point is sofic. These subshifts are known to have applications in geometric group theory [DY02].

[1] A Wang tiling is a finite set of square tiles with markings on each border such that two tiles can be assembled together if they math in the border.

Despite having the property of being a sofic subshift in a wide class of groups we show that this property fails for the class of finitely generated and recursively presented groups which have undecidable word problem, thus we use a computational approach to discover a class of examples which answer an open problem in symbolic dynamics. We also explore a natural generalization of this subshift.

In the second part we explore a class of subshifts which can be understood as tilings of graphs generated by substitutions which can be seen as regular substructures of $\mathbb{Z}^d$. We study the decidability of the emptiness problem, that is, the equivalent formulation of the domino problem in this setting, for such systems with the hope of building a bridge between the rather complicated dynamics of $\mathbb{Z}^2$-subshifts of finite type and their one dimensional counterparts. We also study if those structures can simulate substitutions in the sense of the results by Mozes and Goodman-Strauss [Moz89, GS98] and finally we link those two concepts by means of a theorem which introduces a partial order which monotonically preserves the undecidability of the emptiness problem under the assumption that the bigger substitution satisfies a property analogous to the result by Mozes. Special attention is given to two substitutions which resemble the structures of both the Sierpiński triangle and the Sierpiński carpet.

1. Preliminaries

In this section we define some basic concepts from group theory [Lan02], computation theory [AB09] and symbolic dynamics [LM95, CSC09] which are needed in order to present our results. As these topics and related concepts are extensively covered in the previous references, we stick to short definitions and won't discuss these concepts beyond necessity.

1.1 Group theoretic concepts

Let G be a group and id be its identity element. let $\mathcal{P} = \langle S|R\rangle$ be a presentation of G where S is the set of generators and R the set of relations. The *Cayley graph* of G generated by $\mathcal{P}$ is the directed graph $\Gamma = (G, E)$ where the set of vertices is G and the set of arcs is $E := \{(g, gh) \in G^2 \mid g \in G, h \in S\}$.

Let G and $\mathcal{P} = \langle S|R\rangle$ be a presentation of G. For $g \in G$ we denote $|g|_{\mathcal{P}}$ the function so that $|id|_{\mathcal{P}} = 0$ and $|g|_{\mathcal{P}}$ is the smallest length n of a representation $g = id \cdot g^{(1)} \cdot g^{(2)} \dots g^{(n)}$ so that $\forall 1 \leq i \leq n, g^{(i)} \in S$. If the context is clear we will just write such length as $|g|_G$ or $|g|$. We also define the *ball* of size n, $\Lambda_n = \{g \in G | |g|_{\mathcal{P}} \leq n\}$ which is identified to the portion of the Cayley graph of G generated by $\mathcal{P}$ which contains the identity and every other element at a distance at most n.

A group G is called *finitely generated* if there exists a presentation $\mathcal{P} = \langle S|R\rangle$ of G such that $|S| < \infty$, G is called *finitely presented* if there exists a presentation $\mathcal{P} = \langle S|R\rangle$ where both S and R are finite. Furthermore, a group G is called *recursively presented* if there exists a presentation $\mathcal{P} = \langle S|R\rangle$ where S is countable and R is recursively enumerable, that is, there exists a Turing machine which enumerates every relation in R. It is straightforward to notice that every finitely presented group is both recursively presented and finitely generated, and every recursively presented group where S is finite is finitely generated.

1.2 Computational concepts and the word problem for groups

Let $\mathcal{A}$ be a finite alphabet, we say a language $L \subseteq \mathcal{A}^* = \bigcup_{n\in\mathbb{N}} \mathcal{A}^n$ is *decidable* if there exists a Turing machine such that for every $w \in \mathcal{A}^*$ the machine halts on the input word w, and it accepts w if and only if $w \in L$. We say that L is *recursively enumerable* if there exists a Turing machine that enumerates every element in L, or equivalently, a machine such that if $w \in L$ the machine running on input w halts and accepts, and otherwise it could either halt and reject or it could loop.

Consider a finitely generated group $G = \langle g_1, \dots, g_d | R\rangle$ and g a word in the free monoid M, where $M := \{g_1, \dots, g_d, g_1^{-1}, \dots, g_d^{-1}\}^*$. We want to address the problem of determining whether two different words in that monoid represent the same element in the group. That is, if we denote by $w_1 =_G w_2$ the property that two words seen as products of elements in G are equal, we want to know if the following language is decidable:

$$W := \{g \in M | g =_G id\}.$$

This is known as the *word problem* for groups. Besides the fact that the language above depends on the representation of a group, the word problem for a given set of generators is equivalent to the word problem for another set of generators and thus one can speak unambiguously of the property of having a decidable word problem for a given group. It was shown by Novikov in 1955 [Nov55] that there are finitely presented groups such that the word problem W is undecidable, a more modern and compact proof with a smaller example being given by Collins in 1986 [Col86]. Nonetheless, if the group is recursively presented, then W is recursively enumerable. Indeed, W is the set of words freely equal to a product of conjugates of the set of relations R and this set can be enumerated whenever R can be enumerated.

1.3 Symbolic dynamics concepts

Let $\mathcal{A}$ be a finite alphabet and G a group. We say that the set $\mathcal{A}^G = \{x : G \to \mathcal{A}\}$ equipped with the left group action $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ such that $(\sigma_g(x))_h = x_{g^{-1}h}$ is the *full G-shift*.

By taking the discrete topology in $\mathcal{A}$ we obtain by Tychonoff's theorem that the product topology in $\mathcal{A}^G$ is compact. This topology is generated by a clopen basis given by the cylinders $[a]_g = \{x \in \mathcal{A}^G | x_g = a \in \mathcal{A}\}$. If the group G is countable, then $\mathcal{A}^G$ is metrizable and the compacity of the product topology can be proven directly without using Tychonoff's theorem. In the case of a finitely generated group G, an ultrametric which generates the product topology is given by $d(x, y) = 2^{-\inf\{|g|_G | g\in G : x_g \neq y_g\}}$.

$\mathcal{A}^G$ with the previously defined topology is called a *full G-shift*. A *G-subshift* is a subset $X \subseteq \mathcal{A}^G$ which is topologically closed and invariant under the shift action: $\forall g \in G, \sigma_g(X) = X$.

Let F be a finite subset of G. A *pattern* P is an element of $\mathcal{A}^F$, the set F is called the support or shape of P and is denoted by $supp(P)$. The *set of all patterns*[1] is $\mathcal{A}^* := \bigcup_{F \subseteq G, |F| < \infty} \mathcal{A}^F$. We say that the pattern $P \in \mathcal{A}^{F_1}$ is a *subpattern* of $Q \in \mathcal{A}^{F_2}$ (and we write $P \sqsubseteq Q$) if there exists $g \in G$ such that $gF_1 \subseteq F_2$ and $P = Q_{gF_1}$, we also say that P is a pattern of $x \in \mathcal{A}^G$ (and we write $P \sqsubset x$) if there exists $n \in \mathbb{N}$ such that $P \sqsubseteq x|_{\Lambda_n}$.

A G-subshift can be defined also by a set of forbidden patterns $\mathcal{F} \subseteq \mathcal{A}^*$, that is, $X_{\mathcal{F}} := \{x \in \mathcal{A}^G | \forall g \in G, \forall S \subseteq G : x_{g+S} \notin \mathcal{F}\}$. This definition is equivalent to the previous one as the set of points containing a given pattern can be seen as a cylinder intersection and thus $X_{\mathcal{F}} = \bigcap_{g \in G} \bigcap_{P \in \mathcal{F}} [P]_g^c$ where $[P]_g = \{x \in \mathcal{A}^G | x_{g+supp(P)} = P\}$.

Let X be a G-subshift. We define the set of *globally admissible patterns* of X as:

$$\mathcal{L}(X) = \bigcup_{\substack{F \subseteq G \\ F \text{ finite}}} \mathcal{L}_F(X) = \{P \in \mathcal{A}^* | \exists x \in X : P \sqsubseteq x\}$$

where $\mathcal{L}_F(X) := \{P \in \mathcal{A}^F | \exists x \in X : P \sqsubseteq x\}$ is the set of *globally admissible patterns of shape* F of X.

Let X, Y be two G-subshifts over alphabets $\mathcal{A}_X, \mathcal{A}_Y$ and F a finite subset of G. We say that $\phi : X \to Y$ is a *sliding block code* if there exists a local function $\Phi : \mathcal{A}_X^F \to \mathcal{A}_Y$ such that $\phi(x)_g := \Phi(x_{g+F})$, that is denoted $\phi = \Phi_\infty$. A famous theorem by Curtis, Lyndon and Hedlund [CSC09] identifies the class of sliding block codes with the class of continuous shift commuting functions. We say that a sliding block code ϕ is a *factor code* if it's surjective, and we say it's a *conjugacy* if it's bijective.

Whenever there is a factor code $\phi : X \to Y$ we will write $X \twoheadrightarrow Y$ and say that Y is a *factor* of X and that X is an *extension* of Y. Furthermore, if ϕ is a conjugacy we will write $X \simeq Y$ and say they are *conjugated*. The conjugacy is an equivalence relation, and it preserves most of the topological dynamics of a system.

If $\phi : X \to Y$ is a sliding block code defined by a local function $\Phi : \mathcal{A}_X \to \mathcal{A}_Y$ then we will say that ϕ is a *1-block code*. For every sliding block code $\phi : X \to Y$ it is possible to find a conjugacy $\psi : X \to \hat{X}$ and a 1-block code $\hat{\phi} : \hat{X} \to Y$ such that $\phi = \hat{\phi} \circ \psi$. This means that for every extension of a given G-subshift Y we can ask for a conjugate version $\hat{X}$ of X which extends Y by a 1-block code.

$$\begin{tikzcd}
X \arrow[d, leftrightarrow, "\psi"'] \arrow[dr, "\phi"] & \\
\hat{X} \arrow[r, two heads, "\hat{\phi}"'] & Y
\end{tikzcd}$$

We say that a G-subshift $X \subseteq \mathcal{A}^G$ is a *G-subshift of finite type* (G-SFT) if it can be defined by a finite set of forbidden patterns, that is, $|\mathcal{F}| < \infty$ and $X = X_{\mathcal{F}}$. We say that a G-subshift Y is *sofic* if there

[1] The notation $\mathcal{A}^*$ is the same as the one for the set of words over $\mathcal{A}$ as this generalizes the idea of finite word to finite patterns in a group.

exists a G-SFT X and a factor code ϕ such that $\phi(X) = Y$. The class of sofic G-subshifts is the smallest class closed under factor codes that contains every G-SFT. Both classes are conjugacy invariants, that is, the property of belonging to them is preserved under conjugacy.

Let G be a group with a presentation $\langle S, R\rangle$, we say a G-SFT is *nearest neighbor* if every forbidden pattern $P \in \mathcal{F}$ has a support $\mathit{supp}(P) = \{id, g\}$ where $g \in S$. If G is a finitely generated group, then every G-SFT admits a conjugated version which is nearest neighbor.

2. A class of finitely generated groups where the subshift $X_{\leq 1}$ is not sofic

We have recalled results which say that even when dealing with well understood groups such as $\mathbb{Z}^2$ the properties of sofic subshifts defined over those groups can be quite complicated and are still not well understood. A different way of dealing with the properties of subshifts in the general setting of arbitrary groups is to take a $\mathbb{Z}$-subshift X which is already well understood, generalize its definition to any group G and then pose the following question: for which groups is a dynamical property of X still valid?

For example, consider the even shift in $\mathbb{Z}$, $S_{\text{even}} := \{x \in \{0,1\}^{\mathbb{Z}} | \forall n \in \mathbb{N}_0 : 10^{n+1}1 \not\sqsubseteq x\}$, that is, the subshift which contains bi-infinite sequences which only admit an even number of zeros between two ones. One natural way to generalize this subshift to arbitrary finitely generated groups (the generalization is for any group, but makes sense mainly in this case) is to take a group presentation $\mathcal{P} = \langle S, R\rangle$ and forbid every pattern whose support is a connected component of the Cayley graph of $\mathcal{P}$ such that it contains a connected component of zeros with an odd number of vertices and that component is surrounded by ones in every direction allowed by S. It is easy to show that S_{even} is a sofic $\mathbb{Z}$-subshift, but it is quite interesting the fact that for every finitely generated group G and any presentation of such group given by a finite number of generators, the generalization of S_{even} is a sofic G-subshift too. A proof of this fact is presented in the appendix, see theorem A.1.

While the proof for the case of S_{even} is straightforward, the same definition but with an odd number of zeros, that is, the generalized odd shift S_{odd}, is a much harder case. While the one dimensional case is trivially sofic and the $\mathbb{Z}^2$ case has already been proven to be sofic by Julien Cassaigne (unpublished), The $\mathbb{Z}^d$ case remains an open problem. The main obstruction to making a proof similar to the one given for S_{even} is that it has been proven impossible to realize as a sofic subshift in $\mathbb{Z}^d$ with $d \geq 3$ the subshift in which every finite component has a special marked coordinate, and thus it is impossible to construct a tree with a special marked root in those components. Another reason is that in the even case pasting two components with an even number of zeros yielded another component with the same property, while that is no longer true for odd components.

2.1 The subshift $X_{\leq 1}$

The same question as in the previous section can be asked for subshifts whose generalizations are even easier to describe. Let G be a group and consider the G-subshift defined by:

$$X_{\leq 1} = \{x \in \{0,1\}^G \mid |\{g \in G : x_g = 1\}| \leq 1\}.$$

That is, the set of maps from G to $\{0, 1\}$ where at most one coordinate can be assigned the symbol 1. The motivation for working with this particular subshift arises both for its simplicity (aside from the even shift, it's one of the first examples one would come up with if asked to think about a sofic $\mathbb{Z}$-subshift which is not a $\mathbb{Z}$-SFT) and the fact that it has applications in geometric group theory. In fact, in a paper by Dahmani and Yaman [DY02], some results were proven about the boundary of relatively hyperbolic groups which depend on some groups having the property that $X_{\leq 1}$ is sofic. Surprisingly, there is no complete characterization of the class of groups for which $X_{\leq 1}$ a sofic G-subshift and the existence of any finitely generated group not satisfying the property remained an open problem. Regardless of that, there are a lot of stability results which show that the class is sufficiently large.

Definition 2.1. We will say that G satisfies the *special symbol property* if $X_{\leq 1}$ is a sofic G-subshift.

In the next proposition, we prove that in order to satisfy the special symbol property, a group must be finitely generated. In consequence, from this point forward we only consider groups which fall into that category.

Proposition 2.2. *Let G be a group that satisfies the special symbol property, then G is finitely generated.*

Proof. Let $X \twoheadrightarrow X_{\leq 1}$ be an SFT extension obtained by a 1-block code and defined by a finite set of forbidden patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$. Let $S = \bigcup_{P \in \mathcal{F}} supp(P)$ be the union of the support of every pattern in $\mathcal{F}$, which is finite because it's the finite union of finite sets. We claim that S is a generating set for G. Denote by $\langle S \rangle_G$ the set of elements of G generated by S and suppose by contradiction that there exists $g \in G$ such that $g \notin \langle S \rangle_G$, then $\langle S \rangle_G \cap g\langle S \rangle_G = \emptyset$. Now, take preimages $x^{(id)}, x^{(g)} \in X$ of the point which has exactly a symbol 1 in positions id and g respectively, and consider:

$$x_h = \begin{cases} x_h^{(id)}, & \text{if } h \in \langle S \rangle_G \\ x_h^{(g)}, & \text{if not.} \end{cases}$$

We note that the image of x contains exactly two symbols 1 on positions id and g and thus it doesn't belong to $X_{\leq 1}$, but $x \in X$, because it doesn't contain any forbidden pattern $P \in \mathcal{F}$. If it did, the appearance of P must contain elements from both $\langle S \rangle_G$ and $G \setminus \langle S \rangle_G$ and that is absurd as S contains $supp(P)$. This yields the desired contradiction. $\square$

Proposition 2.3. *The class of finitely generated groups with the special symbol property satisfies the following statements.*

1. *The special symbol property is true for any finite group.*

2. *The special symbol property is true for $\mathbb{Z}$.*

3. *The free group F_k of rank $k \in \mathbb{N}$ satisfies the property.*

4. *The special symbol property is stable under direct products, that is, if $X_{\leq 1}$ is a sofic subshift for the groups G_1 and G_2, then it is also sofic for the group $G_1 \oplus G_2$.*

5. *The special symbol property is satisfied by every finitely generated abelian group.*

The proof for these basic properties can be found in the appendix, see proposition A.2. Other properties which pertain this class of groups are proven in [DY02]. We present the results here just to show that the class is not reduced to the trivial cases presented above.

Proposition 2.4. *The following statements about finitely generated groups are true:*

1. *If a group G splits in a short exact sequence $1 \to N \to G \to H \to 1$ and both N and H satisfy the property, then G also does.*

2. *Let $H \leq G$ be a subgroup with $[G : H] < \infty$, then G has the special symbol property if and only if H has the special symbol property.*

3. *The special symbol property is true for hyperbolic groups.*

4. *Any poly-hyperbolic group satisfies the special symbol property.*

Despite having all these properties, the class of groups satisfying the special symbol property is not the whole class of finitely generated groups. In this section we show a class of finitely generated groups which do not satisfy the property.

Theorem 2.5. *For any group G which is finitely generated, recursively presented and whose word problem is undecidable, the G-subshift $X_{\leq 1} \subseteq \{0,1\}^G$ is not sofic.*

Before starting the proof it is good to state the following remark: if we have two finitely generated groups G_1 and G_2 generated by the same set of generators which are defined by maximal sets R_1 and R_2 of element identifications respectively, that is, sets of pairs $\{g, g'\}$ where g, g' are words in the free monoid of generators that represent the same element in the group, then if $R_1 \subseteq R_2$ and we have defined subshifts over G_1 and G_2 given by an alphabet $\mathcal{A}$ and the same finite set of forbidden patterns $\mathcal{F}$ then every valid tiling of G_2 is a valid tiling of G_1. Indeed, a tiling in any such group G is actually a tiling of the free group (with rank the number of generators of the group) together with the restriction that if $(g =_G g')$ then the positions g and g' must carry the same symbol in every valid tiling.

Proof. We proceed by contradiction by showing that if $X_{\leq 1}$ is sofic then the word problem is decidable. Suppose that $X_{\leq 1} \subseteq \{0,1\}^G$ is sofic, then there exists an SFT extension $\phi : X \twoheadrightarrow X_{\leq 1}$. Furthermore, by choosing a conjugate version of X, one can suppose that $\phi = \Phi_\infty$ is a 1-block code where the special symbol 1 has just one preimage under Φ and (as G is a finitely generated group) X is a nearest neighbor G-SFT. Thus the SFT extension X has a special symbol (which we will also call $1 = \Phi^{-1}(1)$) that can appear at most once in x for every $x \in X$. Let $\mathcal{A}$ be the alphabet of X and $\mathcal{L}(X)$ the set of all globally admissible patterns.

Let G be generated by the set $S := \{g_1, \dots, g_d\}$. The proof relies on the following remark: let $g \in \{g_1, \dots, g_d, g_1^{-1}, \dots, g_d^{-1}\}^*$. We have that $g =_G id \Leftrightarrow \Pi \in \mathcal{L}(X)$, where $\Pi \in \mathcal{A}^{\{id,g\}}$ is the pattern such that $\Pi_{id} = 1$, $\Pi_g = 1$.

We claim that Algorithm 1 decides if $\Pi \in \mathcal{L}(X)$.

Algorithm 1

Input: $\langle S, M, g\rangle$, where S is a finite set, M is a Turing machine which enumerates a set R such that $G = \langle S, R\rangle$ and g is the word for which we want to know whether $g =_G id$.

Output: True if $g =_G id$ and False if $g \neq_G id$.

$T \leftarrow$ R(M) $\triangleright$ T is the machine which enumerates all identifications given by R, that is, generates a maximal list of pairs $\{g_1, g_2\}$ such that $g_1 =_G g_2$.
$Rel, \Theta, \Gamma \leftarrow \emptyset$
$j \leftarrow 0$
while True **do**
  $\Gamma \leftarrow \Lambda_{|g|+j}$ $\triangleright$ $\Lambda_{|g|+j}$ is the ball of size $|g| + j$ of $F_{|S|}$. $\triangleright$ $F_{|S|}$ is the free group on $|S|$ generators.
  $Rel \leftarrow Rel \cup T(j)$ $\triangleright$ $T(j) = \{g_1, g_2\}$ is the j-th output of T.
  if $T(j) = \{id, g\}$ **then**
    return True
  end if
  $\Theta \leftarrow \emptyset$
  for x tiling of Γ by rules of X that respects Rel and Π **do**
    $\Theta \leftarrow \Theta \cup x$
  end for
  if $\Theta = \emptyset$ **then**
    return False
  end if
  $j \leftarrow j + 1$
end while

The algorithm does the following: Let $j = 0$. We iterate the following procedure in a loop: First we construct the directed graph $\Gamma := \Lambda_{|g|+j}$ the ball of size $|g| + j$ of free graph on d generators $S = \{g_1, \dots, g_d\}$ and we run the algorithm T which enumerates every element identification deduced from the set of relations R up to j steps. For every relation between two elements in Γ found (that is, whose word lengths under the free monoid is lesser or equal than $|g| + j$), we add the rule that those two elements in Γ must carry the same symbol under any tiling. After that procedure we find the set Θ of all valid tilings of Γ by using the nearest neighbor rules of X, the element identification rule and with the restriction that $\{id, g\}$ is tiled with the pattern Π. If by iterating the algorithm which enumerates the elements identifications we have obtained that $g =_G id$ then we return that $\Pi \in \mathcal{L}(X)$. If $\Theta = \emptyset$ we return that $\Pi \notin \mathcal{L}(X)$, if none of these happen, we take $j := j + 1$ and we repeat the procedure.

Now we prove that indeed this procedure always stops and returns the correct answer. First note that in every step j the graph Γ has less element identifications that the ball of size $|g| + j$ in the Cayley graph of G, indeed, Γ is an upwards approximation of such ball. In consequence any valid tiling of the ball of size $|g| + j$ of the Cayley graph of G is a valid tiling of Γ.

Suppose $\Pi \in \mathcal{L}(X)$. As the set R is recursively enumerable, we know that in a finite number of steps the algorithm will yield that $g =_G id$. Then Π is actually the pattern which has only one symbol in the identity, and thus it's possible to tile every ball of arbitrary size in the Cayley graph of G such that Π appears as before. Then we can always tile Γ and thus $\Theta \neq \emptyset$ $\forall j \in \mathbb{N}$. In consequence, the procedure won't stop before reaching $g =_G id$ and it will return the correct answer.

Conversely, if $\Pi \notin \mathcal{L}(X)$ there is an integer $N \in \mathbb{N}$ such that the ball of size N of the Cayley graph of G cannot be tiled (if not, by a compacity argument we would deduce that $\Pi \in \mathcal{L}(X)$). Using the fact that this ball is defined by a finite number of element identifications over the ball of the free graph of rank d, and that every one of these identifications will be found after a finite number $M \in \mathbb{N}$ of iterations, one concludes then that after M steps of the algorithm the restriction of Γ to it's ball of size N is exactly the ball of size N of the Cayley graph of G, and thus $\Theta = \emptyset$ in that stage and the process stops and returns the correct answer.

The previous algorithm decides the word problem for G, which is a contradiction given that the group G has unsolvable word problem. □

Corollary 2.6. *For every finitely presented group G with undecidable word problem the G-subshift $X_{\leq 1} \subseteq \{0,1\}^G$ is not sofic.*

Proof. As every finitely presented group if both finitely generated and recursively presented the previous result holds for this class. □

2.2 A generalization of the previous result

A natural generalization of the previous subshift is to allow not only one occurrence of the symbol 1, but a finite number of them. We now show that the same result holds for this class of subshifts.

Definition 2.7. Let G be a group, we define for $k \in \mathbb{N}$

$$X_{\leq k} := \{x \in \{0,1\}^G | \; |\{g \in G : x_g = 1\}| \leq k\}.$$

Let G be a fixed group. It is obvious that if $X_{\leq 1}$ is a sofic subshift then $X_{\leq k}$ is sofic for every $k \in \mathbb{N}$. Just notice that an obvious extension for $X_{\leq k}$ is k copies of $X_{\leq 1}$, that is, $\prod_{i=1}^{k} X_{\leq 1} \twoheadrightarrow X_{\leq k}$ by just projecting any coordinate which has any symbol 1 into the symbol 1 and everything else to 0. By choosing an SFT extension for $X_{\leq 1}$ and taking the product we obtain an SFT extension to $X_{\leq k}$. The other direction is not obvious, and in principle it may happen that there are groups such that $X_{\leq k}$ is sofic but $X_{\leq 1}$ isn't.

We will show that for the case of finitely generated and recursively presented groups with undecidable word problem the subshift $X_{\leq k}$ also cannot be sofic.

Proposition 2.8. *Let G be a finitely generated and recursively presented group. If $X_{\leq k}$ is a sofic G-subshift, then for each set S of $k+1$ words we can decide if either they are all different as elements of G or if there exists a pair that is equal in G.*

Proof. We proceed again by contradiction, suppose $X_{\leq k}$ is sofic and take a nearest neighbor SFT extension X that maps onto $X_{\leq k}$ via a 1-block code ϕ. It is clear (as the alphabet of the extension is finite) that $|\Phi^{-1}(1)| < \infty$. We have that $P \in (\Phi^{-1}(1))^S$ can't belong to $\mathcal{L}(X)$ if the $k+1$ elements of S are different (and thus it's impossible to extend that initial configuration to a tiling of the group). By applying a modified version of the previous algorithm we can decide if either all elements are different in G or if there exists a pair that is equal. The modified algorithm is as follows: instead of searching if $g =_G id$, we search for any equality between members of S, and instead of trying to tile Γ using the pattern Π we try with every pattern $P \in (\Phi^{-1}(1))^S$. The rest of the proof remains the same. $\square$

Theorem 2.9. *Let G be a finitely generated and recursively presented group such that the word problem in G is undecidable, then for each $k \in \mathbb{N}$ $X_{\leq k}$ is not sofic.*

Proof. As the group has undecidable word problem, we know it's infinite. In consequence Λ_k in G has at least $2k+1$ different elements. If we consider sets of the form: $\{id, g_1, \ldots, g_k\}$ with g_i a word in the free group whose length is less than or equal to k. We know there is one of those sets so that each word seen as an element of G is different. By just applying the algorithm from proposition 2.8 over every possible set as above (there are $\binom{d \cdot \frac{(2d-1)^k - 1}{d-1}}{k}$ such sets, where d is the number of generators) we can then obtain $\{id, g_1, \ldots, g_k\}$ so that every word represents a different element in G.

Now consider a word g and the following k sets:

$$\{g\} \cup (\{id, g_1, \ldots, g_k\} \setminus \{g_i\}), \qquad 1 \leq i \leq k$$

There is at least one choice of i such that $g \notin (\{g_1, \ldots, g_k\} \setminus \{g_i\})$ and thus $g \cup (\{g_1, \ldots, g_k\} \setminus \{g_i\})$ has k elements. Now we run in parallel the algorithm which decides if $k+1$ elements are different in G for these k sets. If the algorithm returns for a given set that there are $k+1$ different elements then $g \neq_G id$. If the algorithm returns that there is a repetition for every set it means that for the special case where $g \cup (\{g_1, \ldots, g_k\} \setminus \{g_i\})$ has k elements, adding the identity didn't add an extra element. As the set $\{g_1, \ldots, g_k\}$ doesn't contain the identity we conclude that $g =_G id$. $\square$

3. Tilings of graphs generated by self-similar substitutions

3.1 First definitions

In the beginning of this rapport some results were recalled which show that the emptiness problem is not decidable when considering $\mathbb{Z}^d$-subshifts of finite type with $d > 1$. This result was first proven by Berger [Ber66] who studied the domino problem proposed by Wang and was later on proved using smaller constructions by Robinson [Rob71] and by Kari [Kar96]. Another way to prove this result is by using a rich enough substitution and then applying Mozes theorem, which says that every subshift generated by a $\mathbb{Z}^2$-substitution satisfying a mild consistency condition is of sofic type [Moz89].

In this section we generate classes of $\mathbb{Z}^d$-subshifts which represent graph tilings (that is, sets of functions from graphs to a finite alphabet which respect local rules) which aim to emulate structures lying between $\mathbb{Z}$ and $\mathbb{Z}^d$. We are interested in the study of the emptiness problem in these classes of tilings and in the possibility of these graphs to satisfy a property similar to the one proven by Mozes, that is, the possibility of these graphs to simulate substitutions in themselves by local rules. We first introduce these structures from a symbolic dynamics perspective and then we prove some results concerning the decidability of the emptiness problem and the Mozes-like property, finally we show a result linking these two properties by means of an order relation. In order to define these structures properly and to treat the tilings over them in the language of symbolic dynamics, some definitions are needed.

A *rectangular* $\mathbb{Z}^d$*-substitution* s is a function $s : \mathcal{A} \to \mathcal{A}^R$ where $\mathcal{A}$ is a finite alphabet and R is a *rectangle*, that is, there are natural numbers $l_1, \dots, l_d \geq 2$ such that

$$R := R[l_1, \cdots, l_d] = \{z \in \mathbb{Z}^d | \forall 1 \leq i \leq d,\ 1 \leq z_i \leq l_i\}$$

A substitution can also be regarded as a function from the set of finite patterns $\mathcal{A}^*$ by concatenation of the images. We will assume that every substitution treated from now on is rectangular.

Example 3.1. *Consider the alphabet $\mathcal{A} = \{0, 1\}$ and the $\mathbb{Z}^2$-substitution s such that:*

$$s(0) = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \quad \textit{and} \quad s(1) = \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}.$$

This object is the Sierpiński triangle substitution, and it is the main example we study in this section.

Definition 3.1. A substitution s over the alphabet $\{0, 1\}$ will be said to be a *self-similar* substitution if the image of $s(0)$ is a rectangle containing only the symbol 0.

Any self similar substitution can be defined just by $s(1)$, thus, we can identify the set of self-similar $\mathbb{Z}^d$-substitutions with the set of d-dimensional arrays over $\{0,1\}$, that is, the set of functions from d-dimensional rectangles to $\{0,1\}$. For example, the Sierpiński triangle substitution defined above can be written as $s = \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}$.

Example 3.2. *Consider self-similar $\mathbb{Z}^2$-substitution s such that:*

$$s = \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}.$$

This object is the Sierpiński carpet substitution, and it is another important example.

Definition 3.2. The *subshift generated* by a $\mathbb{Z}^d$-substitution s is the set

$$X_s := \{x \in \mathcal{A}^{\mathbb{Z}^d} | \forall P \sqsubseteq x : P \in \mathcal{L}(s)\}.$$

Where $\mathcal{L}(s) := \{P \in \mathcal{A}^* | \exists a \in \mathcal{A}, \exists n \in \mathbb{N} : P \sqsubseteq s^n(a)\}$ is called the *language* of s.

Theorem 3.3. *Let s be a $\mathbb{Z}^d$-substitution where $d = 2$. Then X_s is a $\mathbb{Z}^d$-subshift of sofic type.*

The above theorem was first proven in the special case of $\mathbb{Z}^2$ by Mozes [Moz89] for a slightly larger set of substitutions where the rules were not needed to be of the same size. The theorem was later largely generalized by Goodman-Strauss [GS98] to a class of $\mathbb{R}^2$ geometrical substitutions satisfying a mild condition where the above theorem in the case of $\mathbb{Z}^2$ sprouts as a corollary. The case of $d > 2$ is seen as a natural generalization of Mozes proof though to the author's knowledge nobody has dared to write it yet.

Definition 3.4. Let s be a $\mathbb{Z}^d$-substitution over the alphabet $\{0,1\}$, let $\mathcal{B} := \mathcal{A} \uplus \{0\}$ a finite alphabet and $\mathcal{F} \subseteq \mathcal{B}^*$ a finite set of forbidden patterns. Define the *projection sliding block code* as the map $\pi : \mathcal{B}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ by $\pi = \Pi_\infty$ where $\Pi(0) = 0$ and $\forall a \in \mathcal{A}$, $\Pi(a) = 1$. with these elements in hand we define the *set of tilings of s* given by $\mathcal{F}$ as the $\mathbb{Z}^d$-subshift:

$$X^s_{\mathcal{F}} := \{x \in \mathcal{B}^{\mathbb{Z}^d} | \pi(x) \in X_s, \forall P \sqsubseteq x : P \notin \mathcal{F}\}$$

For simplicity, we restrict the study to the case where s is a self-similar substitution and unless said otherwise we suppose that for every $P \in \mathcal{F}$ then *supp*$(P) \subseteq U$ with $U = \{-1,0,1\}^d$. Also, in order to rule out the trivial empty case, we demand that no block consisting only on zeros is forbidden. Thus every tiling of a self-similar substitution s will always contain the point 0^∞ consisting only on zeros.

The subshift $X^s_{\mathcal{F}}$ when $\mathcal{F}$ satisfies the above conditions can be interpreted as the set of tilings of the limit graph generated by iterating the substitution s over the symbol 1, and then identifying every 1 with a vertex and putting an edge between every pair of adjacent (according to the shape of the patterns in $\mathcal{F}$)

ones in the resulting matrix. Figure 3.1 shows this construction for the substitution given in example 3.1 and a set $\mathcal{F}$ with support in $U = \{-1, 0, 1\}^2$.

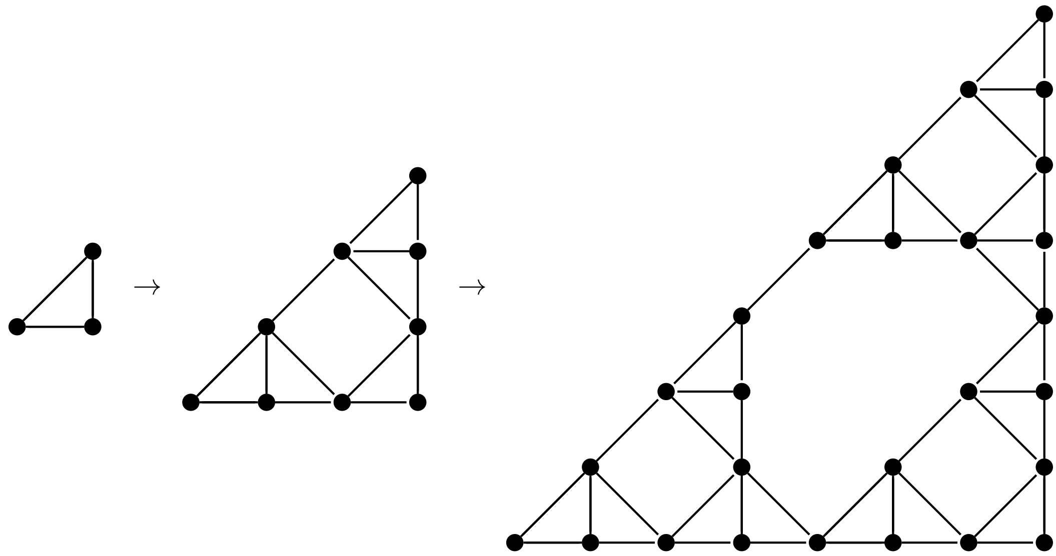


FIGURE 3.1: The Sierpiński triangle substitution as a graph substitution.

The formalization given above lets us consider colorings of such graphs while avoiding the need to give a definition of what is a limit graph. It also helps to avoid the problem of defining the local rules in the graph setting and it allows the tiling to use the border information in order to enforce local rules.

By using theorem 3.3 we obtain that X_s is a sofic shift and has an SFT extension X. Considering $X \times (X_{\mathcal{F}})$ with $\mathcal{F} \subseteq \mathcal{A}^*$ along with the extra rules that in X a symbol which is projected into a 0 must carry a 0 in a second coordinate and the rest must carry a symbol from $\mathcal{A}$, then we obtain that $X^s_{\mathcal{F}}$ is also a sofic $\mathbb{Z}^d$-subshift by just projecting the previous construction onto the second coordinate.

3.2 The emptiness problem and the Mozes property

Let s be a fixed self-similar substitution, and let L be the set of all $\mathcal{F} \subseteq \mathbb{N}^*$ such that $X^s_{\mathcal{F}} \neq \{0^\infty\}$, that is, the set of all sets of forbidden patterns over a finite alphabet[1] such that the set of tilings of s is non-empty (does not restrict itself to the trivial point containing only zeros). A natural question is for which self-similar substitutions s is L decidable. This is the formulation of the *emptiness problem* in this particular setting.

If we consider s to be the array containing only zeros then the problem above is clearly decidable, in the other hand, if s is the array containing only ones, the set of possible tilings corresponds to the class of $\mathbb{Z}^2$-SFTs, and thus the problem above is undecidable as a consequence of theorem 3.3.

Beside the trivial examples shown above, we show intermediate cases of self-similar $\mathbb{Z}^2$-substitutions which fall into these two categories.

[1]Here we identify a finite alphabet $\mathcal{A}$ with cardinality n to the set $\{1, 2, \ldots n\}$.

Theorem 3.5. *The Sierpiński triangle substitution from example 3.1 has decidable emptiness problem.*

Proof. Consider $n \in \mathbb{N}$ and the set of all possible tilings of $s^n(1)$, that is, the set of patterns in $\mathcal{A}^{2^n \times 2^n} \cap \mathcal{L}(X^s_{\mathcal{F}})$ such that their image under the projection sliding block code π is $s^n(1)$. Notice that as $s^{n+1}(1) = s^n(s(1))$ and s is self-similar, then $s^{n+1}(1)$ is constructed by pasting together three pieces of $s^n(1)$ along with a block of zeros. It is easy to proof inductively that for each $n \in \mathbb{N}$ $s^n(1)$ has no ones over the diagonal, and thus in order to construct any tiling over $s^{n+1}(1)$ it suffices to paste three tilings of $s^n(1)$ which respect the local rules $\mathcal{F}$ in the point where they meet, that is, the only points relevant in order to decide if three tilings of $s^n(1)$ can be pasted in order to make a tiling of $s^{n+1}(1)$ are the following (suppose $s^n(1)$ has support $R[2^n, 2^n]$):

$$C = \{(1,1), (1, 2^n - 1), (1, 2^n), (2, 2^n), (2^n, 2^n)\}.$$

With that information, it is straightforward to construct an algorithm for deciding the emptiness problem: given $\mathcal{A}$ and $\mathcal{F}$ with support in U construct all possible tilings of $s^2(1)$ and for every one of them store the image of the set C as a tuple in $\mathcal{A}^5$. That gives an element of $P(\mathcal{A}^5)$ which is finite. As the tilings of $s^{n+1}(1)$ depend on the tilings of $s^n(1)$, then also the attainable tuples of $s^{n+1}(1)$ depend only on the tuples attainable by $s^n(1)$, we continue to store this elements in a list. If there is a repetition of a non-empty element of $P(\mathcal{A}^5)$, then as the adjacencies just depend on the previous configuration, the sequence of attainable tuples becomes eventually periodic and therefore it is possible to construct arbitrary big tilings and the algorithm returns that $X^s_{\mathcal{F}} \neq \{0^\infty\}$. Either this must happen before $2^{\mathcal{A}^5}$ iterations of the algorithm or a big enough triangle will not be able to be tiled (that is, we obtain $\emptyset \in P(\mathcal{A}^5)$ as the set of configurations for a big enough n). In that case the algorithm returns that $X^s_{\mathcal{F}} = \{0^\infty\}$ $\square$

The above proof can be easily generalized to any self-similar substitution where the number of new adjacent ones (if seen as a graph, the number of new edges) obtained by pasting pieces of $s^n(1)$ to form $s^{n+1}(1)$ is bounded. In particular a countable class of 2-dimensional self-similar substitutions which can be proven to be decidable by the above argument is the Pascal triangle modulo $m \geq 2$, that is, $s_m \in \{0,1\}^{R[m,m]}$ where $s_{(i,j)} = 1 \Leftrightarrow i \leq j$.

Before presenting the undecidability results it is convenient to introduce a related concept, which determines the possibility that a self-similar substitution is able to simulate more complex substitutions as tilings of its structure.

Definition 3.6. Let s be a self similar substitution defined over the rectangle R. We say that a substitution s' over a finite alphabet $\mathcal{A} \cup 0$ and defined over the same rectangle R is an *X_s-substitution* if $s'(0) = 0^R$ and for every $a \in \mathcal{A}$ and position $p \in R$: $s'(a)_p = 0$ if and only if $s(1)_p = 0$.

Definition 3.7. We say that a self similar substitution s satisfies the *Mozes property* if for every X_s-substitution s' there exists an alphabet $\mathcal{A}'$ and a set of local rules $\mathcal{F}' \subseteq (\mathcal{A}' \cup \{0\})^*$ and a local function $\Phi : \mathcal{A}' \to \mathcal{A}$ such that $\phi(X^s_{\mathcal{F}'}) = X_{s'}$, where $\phi = \Phi_\infty$.

This property is the equivalent of Mozes theorem for substitutions defined over self-similar substitutions. It may seem redundant at first glance, but the fact that all X_s-substitutions are sofic $\mathbb{Z}^d$-subshifts does not mean that all substitutions s satisfy the Mozes property, for example consider the diagonal substitution $s = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$.

The subshift generated by s consists on the shift closure of the point consisting of a single diagonal of ones immersed in a sea of zeros. If we allow the forbidden patterns to take support in U as usual, then there is an obvious correspondence between the set of tilings of s and the class of 1-step $\mathbb{Z}$-SFT[2]. On the other hand, if we consider the X_s-substitution s' given by:

$$a \rightarrow \begin{matrix} 0 & b \\ a & 0 \end{matrix}, \quad b \rightarrow \begin{matrix} 0 & a \\ b & 0 \end{matrix}$$

then the $\mathbb{Z}$-subshift which is in correspondence to the tiling of the diagonal is the one generated by the Thue-Morse sequence (see [FB02] for this fact), which is not of sofic type, and thus can not be obtained as the image by a sliding block code of an SFT.

Next we show that the two examples shown at the beginning of this section satisfy the Mozes property.

Theorem 3.8. *The Sierpiński triangle from example 3.1 satisfies the Mozes property.*

Theorem 3.9. *The Sierpiński carpet from example 3.2 satisfies the Mozes property.*

The proof for the previous theorems are a little long, they can be found on the appendix, see theorems A.3 and A.4.

With these two results in hand we proceed to prove that the Sierpiński carpet substitution has undecidable emptiness problem. The proof will consist on the simulation of a substitution which allows arbitrarily big square blocks of $\mathbb{Z}^2$ to be simulated in the structure.

Theorem 3.10. *The Sierpiński carpet substitution from example 3.2 has undecidable emptiness problem.*

Proof. Let s be the Sierpiński carpet substitution and consider s' the X_s-substitution over the alphabet $\mathcal{A} = \{\bullet, \boxtimes, \boxdot, \updownarrow, \leftrightarrow\}$ given by the following rules:

$$\bullet \rightarrow \begin{matrix} \boxtimes & \leftrightarrow & \boxtimes \\ \updownarrow & & \updownarrow \\ \boxtimes & \leftrightarrow & \boxdot \end{matrix}, \quad \boxtimes \rightarrow \begin{matrix} \boxtimes & \leftrightarrow & \boxtimes \\ \updownarrow & & \updownarrow \\ \boxtimes & \leftrightarrow & \boxtimes \end{matrix}, \quad \boxdot \rightarrow \begin{matrix} \boxtimes & \leftrightarrow & \boxtimes \\ \updownarrow & & \updownarrow \\ \boxdot & \leftrightarrow & \boxtimes \end{matrix}$$

$$\updownarrow \rightarrow \begin{matrix} \updownarrow & \bullet & \updownarrow \\ \updownarrow & & \updownarrow \\ \updownarrow & \bullet & \updownarrow \end{matrix}, \quad \leftrightarrow \rightarrow \begin{matrix} \leftrightarrow & \leftrightarrow & \leftrightarrow \\ \bullet & & \bullet \\ \leftrightarrow & \leftrightarrow & \leftrightarrow \end{matrix}.$$

[2] A $\mathbb{Z}$-SFT is said to be n-step if it can be defined with a set of forbidden words of length at most $n+1$.

As s satisfies the Mozes property there exists a set of tilings $X^s_{\mathcal{F}}$ and a 1-block factor code ϕ such that $\phi : X^s_{\mathcal{F}} \twoheadrightarrow X_{s'}$. We claim that in any point of $X_{s'}$ which is not 0^∞ for any $n \in \mathbb{N}$ the pattern $s'^n(\bullet)$ appears infinitely often. In order to prove that, it suffices to use the fact that any tiling of $s^n(1)$ is $s'^n(a)$ for $a \in \mathcal{A}$. Checking in the rules above that regardless of the starting symbol a, $\bullet$ appears at most after 2 iterations of s', it suffices to consider $s'^{n+2}(a)$ to be sure to find $s'^n(\bullet)$ as a subpattern. As we can do this for every block $s'^{n+2}(a)$ we obtain an infinite number of occurrences of such a block.

Now consider the block $s'^n(\bullet)$ for $n \geq 1$. It is easy enough to see by means of an induction argument that this will consist on a lattice of $2^n \times 2^n$ boxes which are being connected by arrows as in figure 3.2, along with smaller instances $s'^m(\bullet)$ for $m \leq n-2$. Of all of these boxes in the lattice, the only box $\boxdot$ will be the one in position $(1, 2^{n-1}+1)$ of the lattice.

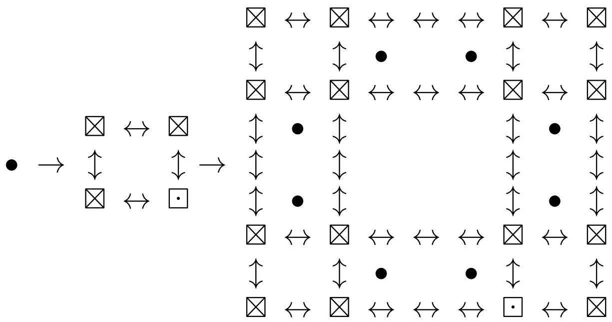
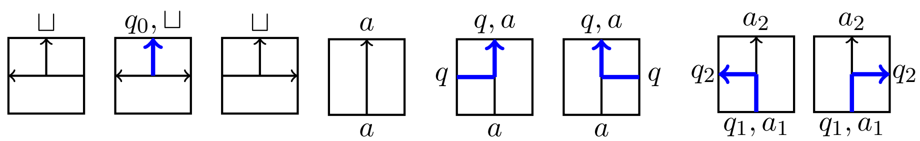


FIGURE 3.2: The three first levels of $s'^n(\bullet)$.

The previous lattice can be used to simulate patterns of size $2^n \times 2^n$ of a nearest neighbor $\mathbb{Z}^2$-SFT $X_{\mathcal{G}}$ in a second coordinate. It suffices to let any symbol of $\mathcal{A}$ carry any symbol appearing in $X_{\mathcal{G}}$ in a second coordinate with the restrictions that two horizontally adjacent horizontal arrows must carry the same symbol, two vertically adjacent vertical arrows must carry the same symbol and given a box tile, if there is a horizontal arrow to the right or a vertical arrow down, then the symbols on the second coordinate must not be forbidden in $\mathcal{G}$. If there is a horizontal arrow to the left or a vertical arrow up then they must share the same symbol in the second coordinate. In this way the $2^n \times 2^n$ symbols in the lattice of boxes correspond to the coordinates of a square pattern of size $2^n \times 2^n$ in $X_{\mathcal{G}}$. An example can be seen in figure 3.3.

Now consider the alphabet given by a set of tiles which simulates a Turing machine $M = \langle Q, q_0, \Gamma, \sqcup, F, \delta \rangle$[3] running on a empty input.



Here the three first tiles define a Turing machine starting on a empty input in the initial state q_0, the symbols a and q denote an element of Γ and Q respectively and finally the two last tiles represent

[3]Where Q is the set of states, q_0 the initial state, Γ is both the tape and starting alphabet, $\sqcup \in \Gamma$ the blank symbol, F the set of final states and $\delta : Q \setminus F \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function

$$
\begin{array}{cccc}
a_2 & a_2 & a_3 & a_4 \\
b_2 & b_2 & b_3 & b_4 \\
c_2 & c_2 & c_3 & c_4 \\
d_2 & d_2 & d_3 & d_4
\end{array}
\rightarrow
\begin{array}{ccccccccc}
(\boxtimes, a_1) & (\leftrightarrow, a_2) & (\boxtimes, a_2) & (\leftrightarrow, a_3) & (\leftrightarrow, a_3) & (\leftrightarrow, a_3) & (\boxtimes, a_3) & (\leftrightarrow, a_4) & (\boxtimes, a_4) \\
(\updownarrow, b_1) & & (\updownarrow, b_2) & (\bullet, \cdot) & & (\bullet, \cdot) & (\updownarrow, b_3) & & (\updownarrow, b_4) \\
(\boxtimes, b_1) & (\leftrightarrow, b_2) & (\boxtimes, b_2) & (\leftrightarrow, b_3) & (\leftrightarrow, b_3) & (\leftrightarrow, b_3) & (\boxtimes, b_3) & (\leftrightarrow, b_4) & (\boxtimes, b_4) \\
(\updownarrow, c_1) & (\bullet, \cdot) & (\updownarrow, c_2) & & & & (\updownarrow, c_3) & (\bullet, \cdot) & (\updownarrow, c_4) \\
(\updownarrow, c_1) & & (\updownarrow, c_2) & & & & (\updownarrow, c_3) & & (\updownarrow, c_4) \\
(\updownarrow, c_1) & (\bullet, \cdot) & (\updownarrow, c_2) & & & & (\updownarrow, c_3) & (\bullet, \cdot) & (\updownarrow, c_4) \\
(\boxtimes, c_1) & (\leftrightarrow, c_2) & (\boxtimes, c_2) & (\leftrightarrow, c_3) & (\leftrightarrow, c_3) & (\leftrightarrow, c_3) & (\boxtimes, c_3) & (\leftrightarrow, c_4) & (\boxtimes, c_4) \\
(\updownarrow, d_1) & & (\updownarrow, d_2) & (\bullet, \cdot) & & (\bullet, \cdot) & (\updownarrow, d_3) & & (\updownarrow, d_4) \\
(\boxtimes, d_1) & (\leftrightarrow, d_2) & (\boxtimes, d_2) & (\leftrightarrow, d_3) & (\leftrightarrow, d_3) & (\leftrightarrow, d_3) & (\boxdot, d_3) & (\leftrightarrow, d_4) & (\boxtimes, d_4)
\end{array}
$$

FIGURE 3.3: The simulation of the 4×4 pattern in the left in the lattice. The bullets can carry any symbol from the alphabet of $X_{\mathcal{G}}$.

transitions from δ. $\delta(q_1, a_2) = (q_2, a_2, L)$ and $\delta(q_1, a_2) = (q_2, a_2, R)$ respectively. The nearest neighbor rules for these tiles are just that same color arrow tails must match with arrow heads, and in the case any of those carry information, they must match. This simulates a Turing machine running on a empty input, each line representing a step in the computation while holding all the information of the tape[4]. The important thing to notice in this construction is that if we are given an infinite line constructed by using the first three tiles and in which the second tile (the starting point of the tape) appears, then the tiles given above are able to tile the upper semiplane starting from that line if and only if the machine M never halts.

By using the extension $X^s_{\mathcal{F}}$ to simulate X_s and then the rules described above to simulate the tiling given by the Turing machine M (that is, every rule which involved a symbol in $\mathcal{A}$ now involves every symbol in it's preimage via the 1-block code ϕ) along with the extra rule that every $\boxdot$ must carry the starting computation symbol we obtain a finite alphabet $\mathcal{B}$ along with a finite set of forbidden patterns $\mathcal{H}$ such that $(\mathcal{B}, \mathcal{H})$ tiles the Sierpiński carpet if and only if the Turing machine M loops in the empty input. Suppose the emptiness problem was decidable for the Sierpiński carpet, then we would obtain a decision algorithm to see if a Turing machine halts on empty input, which yields a contradiction. $\square$

Notice that the above proof can be modified in order to make a proof of the undecidability of the emptiness problem for $\mathbb{Z}^2$. Just consider the same substitution s' as above with the modification such that $s'(a)_{(2,2)} = \bullet$ for every $a \in \mathcal{A}$. Theorem 3.3 ensures this primitive substitution generates a sofic $\mathbb{Z}^2$-subshift and the same proof as above can be applied.

The proofs given above for the Sierpiński carpet can be easily generalized to a countable class of $\mathbb{Z}^2$-substitutions which resemble the carpet. For example, take $n \in \mathbb{N}$ and pairs (i, i') and (j', j') such that $i < i' - 1$, $j' < j' - 1$ and they lie between 1 and n. Consider the $\mathbb{Z}^2$-substitution s such that $s_{(k,l)} = 1 \Leftrightarrow k \in i, i' \vee l \in \{j, j'\}$. The tuples for the Mozes property construction can be forced in position (i, j) and arbitrarily big lattices can be simulated via substitutions in a similar way as in the proof of the last theorem.

[4] Similar constructions can be found all over the literature, see for example [Rob71].

Next we show the first general theorem relating the Mozes property and the decidability of the emptiness problem. Before stating the theorem we must introduce a partial order relation between substitutions with the same support.

3.3 A partial order preserving decidability monotonically

Definition 3.11. Let s_1 and s_2 be two self-similar substitutions defined over the same rectangle. We say that $s_1 \preceq s_2$ if and only if for every array position p we have that:

$$(s_2)_p = 0 \Rightarrow (s_1)_p = 0.$$

Theorem 3.12. *Let s_1 and s_2 be two self-similar substitutions such that s_2 satisfies the Mozes property and $s_1 \preceq s_2$, then if s_2 has a decidable emptiness problem then s_1 has a decidable emptiness problem as well.*

Proof. First note that for each $n \in \mathbb{N}$ if we define $s_1^n := s_1^n(1)$ then $s_1^n \preceq s_2^n$ and $X_{s_j} = X_{s_j^n}$ for $j = 1, 2$ thus the decidability of the emptiness problem is equivalent if we consider the power of a substitution. Suppose the substitutions are defined over the rectangle $R[l_1, \dots, l_d]$. For $S \subseteq \mathbb{Z}^d$ define the *diameter* of S as $diam(S) := \max_{g,h \in S} \max_{1 \leq i \leq d} \{|g_i - h_i|\}$. And for a set of forbidden patterns $\mathcal{F}$ let $diam(\mathcal{F}) := \max_{P \in \mathcal{F}} diam(supp(P))$.

Let $(\mathcal{A}, \mathcal{F})$ an alphabet and a finite set of forbidden patterns. First we rule out the trivial case where $X_{s_2} = \{0^\infty\}$ because in that case the result is trivially true. Consider then $m \in \mathbb{N}$ and $\bar{p} \in R[l_1^m, \dots, l_d^m]$ such that $(s_2^m)_{\bar{p}} = 1$ and $\bar{p} + \Lambda_{2 \cdot diam(\mathcal{F})+1} \subseteq R[l_1^m, \dots, l_d^m]$. Such values must always exist because on the contrary we would have that $X_{s_2} = \{0^\infty\}$.

Consider the $X_{s_2^m}$-substitution s' over the alphabet $\mathcal{B} = \{1, *\}$ given by the rules:

$$s'(1)_p = \begin{cases} 0, & \text{if } (s_2^m)_p = 0 \\ 1, & \text{if } (s_1^m)_p = 1 \\ *, & \text{if } (s_1^m)_p = 0 \wedge (s_2^m)_p = 1 \end{cases} \quad , \quad s'(*)_p = \begin{cases} 0, & \text{if } (s_2^m)_p = 0 \\ 1, & \text{if } p = \bar{p} \\ *, & \text{if } (s_1^m)_p = 1 \wedge p \neq \bar{p}. \end{cases}$$

The above substitution emulates s_1^m in the structure of s_2^m. In fact, if we consider a simplified substitution where the symbol $*$ is substituted into a block of 0 and $*$ then by identifying 0 and $*$ the structure of ones arising from a 1 is exactly $s_1^m(1)$. Nevertheless, in order to forbid infinite points of $*$ appearing and thus rendering the decidability problem always true, it is necessary to introduce new seeds of 1 which force the blocks $(s_1^m)^n(1)$ to appear infinitely often for all $n \in \mathbb{N}$ and in any point of $X_{s'}$. The idea of putting the extra seed of 1 in a position $\bar{p}$ such that $\bar{p} + \Lambda_{2 \cdot diam(\mathcal{F})+1} \subseteq R[l_1^m, \dots, l_d^m]$ is that every new appearance of 1 (and block derived by further substituting this) is sufficiently far from the rest of the ones in the array, and thus the structures can be tiled independently without forbidden patterns appearing between the two structures.

Consider a set of tilings Y with two layers so that the first layer is an extension given by a 1-block code $\phi = \Phi_\infty$ such that $\phi : X^{s_2^m}_{\mathcal{F}'} \twoheadrightarrow X_{s'}$ and the second layer is given by the set of tilings of s_2^m with the alphabet $\mathcal{A} \cup \{\bullet\}$ and the forbidden patterns $\mathcal{H}$ where $\mathcal{H}$, is a copy of the patterns in $\mathcal{F}$ where any number of zeros can be replaced by $\bullet$. We finally add the linking rule between the two layers such that any symbol in the first coordinate which projects via Φ to 1 must carry a symbol from $\mathcal{A}$ in the second coordinate and every $*$ must have $\bullet$ in the second coordinate. We claim that $Y \neq \{0^\infty\}$ if and only if $X^{s_1^m}_{\mathcal{F}} \neq \{0^\infty\}$. This will finish the proof as there is a decision algorithm to know whether $Y \neq \{0^\infty\}$.

Suppose first that $X^{s_1}_{\mathcal{F}} = \{0^\infty\}$. As we discarded the trivial case where $X_{s_1} = \{0^\infty\}$ there must be $n \in \mathbb{N}$ such that $s_1^{mn}(1)$ can not be tiled by $(\mathcal{A}, \mathcal{F})$. As $s'(1)_p = 1$ if $s_1^m(1)_p = 1$ then $s'^n(1)_p = 1$ if $s_1^{mn}(1)_p = 1$ and as this component of ones is surrounded by the symbols 0 and $*$ at least in a radius of $2 \cdot diam(\mathcal{F}) + 1$ we conclude that there are not valid tilings in the second coordinate of Y which tile $s_2^{mn}(1)$.

Now suppose that $X^{s_1}_{\mathcal{F}} \neq \{0^\infty\}$. From this we know that there are valid tilings of $s_1^n(1)$ for every $n \in \mathbb{N}$. We construct a tiling for the second coordinate of Y restricted to $s_2^{nm}(1)$, that is, we tile each component of ones in $s'^n(1)$ which coincides with the positions of the ones in $s^{mj}(1)$ for $j = n, n-1, \dots, 1$. The rest can be filled with $\bullet$. As the distance between each component of ones arising directly from a 1 (that is, that in its substitution history didn't came from a $*$) is at least $2 \cdot diam(\mathcal{F}) + 1$, we conclude that no forbidden patterns were formed and this is indeed a valid tiling of $s_2^{nm}(1)$ which has $s'^n(1)$ in the first coordinate. We conclude by compacity that $Y \neq \{0^\infty\}$. □

Corollary 3.13. *Let s_1 and s_2 be two self-similar substitutions such that $s_1 \preceq s_2$ and s_2 satisfies the Mozes property, then if s_1 has an undecidable emptiness problem then s_2 has also an undecidable emptiness problem.*

Yet another proof of the fact that emptiness problem is undecidable on $\mathbb{Z}^2$ can be given using the above corollary. On one hand we have that the set of all $\mathbb{Z}^2$-SFT correspond to the set of tilings of the substitution $s_2 = 1^{R[3,3]}$ and by theorem 3.3 that substitution satisfies the Mozes property. In the other hand we can take s_1 as the Sierpiński carpet substitution and $s_1 \preceq s_2$. Thus the result follows from corollary 3.13 and theorem 3.10.

Conclusions

In the first part of this rapport we presented a countable class of groups for which the subshift $X_{\leq 1}$ is not of sofic type and we generalized the construction in order to prove the same result for $X_{\leq k}$ with $k \in \mathbb{N}$. Nevertheless we are still far from a total characterization of the set of groups which satisfy the special symbol property as defined in 2.1. In particular, we believe that certain groups which have a lot of cycles and an intermediate growth of the border $\partial\Lambda_n := \Lambda_n \setminus \Lambda_{n-1}$ could also serve as examples for which the special symbol property is not held. In particular, an interesting group which answers to both the Burnside [ALW11] and the Milnor problems [Gri11] (that is, every element has finite order and the border growth is intermediate between polynomial and exponential) is the Grigorchuk group [Gri84], which we believe is a good candidate of a group with decidable word problem so that it may not satisfy the special symbol property. Also the Lamplighter group [Nek05] has been suggested by Dahmani as a potential candidate.

In the second part of this work we presented a formal framework in order to simulate tilings of graphs generated by self-similar substitutions. The study of this class of substitutions was motivated by the idea of building a bridge between the undecidability of the emptiness problem in $\mathbb{Z}^2$ and the decidability of its counterpart in $\mathbb{Z}$. We started this study with the naive idea that the Hausdorff dimension of a fractal which could be simulated by a self-similar substitution s, that is, the box counting dimension of a transitive point of X_s, could hold the key to determining the decidability of the emptiness problem. Nevertheless, using an argument similar to the one presented in theorem 3.5 by filling the center of a substitution with ones while leaving zeros in the border we obtain that there are substitutions with arbitrarily big dimension and decidable emptiness problem, thus showing that the initial idea of a dimension threshold didn't hold. We then proceeded to show examples which satisfied the emptiness property and some who didn't while linking this to the Mozes property we defined in 3.7. The reader can note that while the hierarchical layers in the two proofs regarding the Mozes property were different (in the triangle case a very light structure sufficed while in the carpet a more robust anatomy was necessary) the proof was essentially the same. We believe the proof can be generalized to wider classes of self-similar substitutions whose graphs satisfy a connexity condition.

Conjecture 4.1. *let s be a self-similar substitution such that for every $n \in \mathbb{N}$ the graph defined by $s^n(1)$ and the adjacencies in $U = \{-1,0,1\}^d$ is 2-connected [Die06], then s satisfies the Mozes property.*

If this conjecture above is true, it may be possible to characterize completely which self-similar substitutions satisfy the Mozes property and thus that can be used to determine the decidability of the emptiness problem for larger classes using theorem 3.12. In particular, given a fixed rectangle R, it may be useful to determine which is the boundary between decidability and undecidability in the lattice given by the poset $(\{0,1\}^R, \preceq)$ in view of theorem 3.12. Though still some examples would remain for which we haven't been able to classify its decidability, see the following table.

Decidable substitutions	Unknown decidability	Undecidable substitutions
Substitutions for which the proof of theorem 3.5 can be adapted	?	Substitutions where arbitrarily big lattices can be simulated by a substitution as in 3.10.
Example: $\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}$	Example: $\begin{matrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}$	Example: $\begin{matrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{matrix}$

Appendix

In this section we present proofs of either claims given in the previous sections which are not strictly necessary to the topics presented in this report but that require some sort of justification that is not in the literature, or proofs of results that are simply too long to fit in the 20 page limit.

Theorem A.1. *For every finitely generated group G and any presentation of such group given by a finite number of generators, the generalization of S_{even} as defined in Section 1.3 is a sofic G-subshift.*

Proof. The proof will proceed by construction of an extension X and a 1-block code $\phi : X \twoheadrightarrow S_{\text{even}}$. Suppose that the group G is presented by $\mathcal{P} = \langle S, R \rangle$ and take the set $K = \Lambda_1 \setminus \{id\}$, that is, the set of generators S along with their inverses so that every element identification arising from R has already been applied. If G is already a finite group the proof is trivial, if not, then K has at least two elements. Now, consider the finite set of functions $(\varphi_i)_{i \in I}$ with $\varphi_i : K \to \{0, 1\}$ such that $|\varphi_i^{-1}(1)|$ is either 0 or an odd number (an example can be seen in figure A.1). Now let $\mathcal{A} = (\varphi_i)_{i \in I}$ be a finite alphabet and consider the subshift $X \subseteq \mathcal{A}^G$ given by the finite amount of rules such that for every $g \in K$ the block P with support $\{id, g\}$ is forbidden if $P_{id}(g) \neq P_g(g^{-1})$, that is, if they don't match up as Wang tiles. Consider also the 1-block code Φ that maps $\varphi_i \equiv 0$ to 1 and any other function to 0. and take $\phi = \Phi_\infty$. We claim that X is an extension mapping onto S_{even} via ϕ.

First we show that for every $x \in X$ the image lies in S_{even}. Take an $x \in X$ and suppose there is a finite maximal component of zeros in $\phi(x)$. By definition of ϕ the preimage x must have a finite connected component of functions not equal to 0 surrounded by $\varphi \equiv 0$. Identify each of these nonzero functions as nodes in a finite graph with edges between two nodes whenever they are adjacent in the Cayley graph and they are matched by the symbol 1 in the direction of adjacency. We see that every node in this graph has an odd degree as every φ was defined in this way. By the handshaking lemma we obtain that $\sum \deg(v) = 2|E|$. As the degree of every vertex is odd, we conclude that the number of vertices must be even. Thus every maximal finite connected component of 0s in $\phi(x)$ must be even.

Now we show that the function is surjective. Take $y \in S_{\text{even}}$. Obviously every symbol 1 must have as preimage a 0, so we only have to deal with the preimage of both finite and infinite maximal components of 0s. Take first a finite maximal even component of 0s and consider the underlying portion of the Cayley graph it defines. Take a covering tree of such component. If every node in the covering tree has odd degree then this can be realized as elements of $\mathcal{A}$ and we are done, if not, take a vertex in the covering tree that has even degree and consider the subtrees generated by deleting this vertex. As the number of vertices in the tree is even, there must exist an odd number of subtrees that have odd degree. By reconnecting this vertex to every such odd component (we use a symbol from $\mathcal{A}$), we obtain a forest of subtrees where each has an even number of vertices and the total number of vertices with even degree has been reduced by one. Iterating this procedure yields a forest where every node has odd degree and thus a covering of the component that can be realized by elements of $\mathcal{A}$. For the case of infinite components we proceed by a compacity argument. Using the previous procedure we generate a sequence of points $(x_n)_{n\in\mathbb{N}}$ where $x_n \in \mathcal{A}^G$ and such that $\phi(x_n)|_{\Lambda_{n-1}} = y|_{\Lambda_{n-1}}$ and $x_n|_{\Lambda_n}$ respects every rule of X (we can do so by modifying $y|_{\Lambda_n}$ by deleting a 0 in the border $\Lambda_n \setminus \Lambda_{n-1}$ in each component with an odd number of zeros and then taking a preimage using the previous construction). By compacity we extract a converging subsequence $x_{n_\alpha} \to \bar{x}$. Clearly $\bar{x} \in X$ and by continuity of ϕ we obtain that $\phi(x) = y$.

□

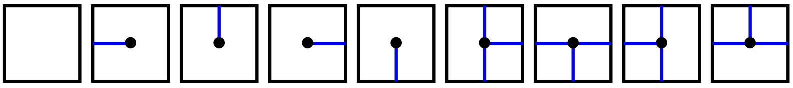


FIGURE A.1: The alphabet $\mathcal{A}$ in the extension of S_{even} for $\mathbb{Z}^2$ represented as Wang tiles.

Proposition A.2. *The properties from proposition 2.3 are true.*

Proof. 1) The property is trivial for finite groups because any subshift defined over a such a group is actually an SFT.

2) For $G = \mathbb{Z}$ consider the alphabet $\mathcal{A} = \{\leftarrow, 1, \rightarrow\}$ and the set of forbidden patterns $\mathcal{F} := \{\leftarrow\rightarrow, 1\leftarrow, 11, \rightarrow\leftarrow, \rightarrow 1\}$. The 1-block code given by the local rule $\phi = \Phi_\infty$ that sends $\Phi(1) = 1$ and $\Phi(\leftarrow) = \Phi(\rightarrow) = 0$ is a factor code onto $X_{\leq 1}$.

$$\begin{array}{rl} x = & \cdots \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \; 1 \; \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \cdots \\ & \qquad\qquad\quad \Big\downarrow \phi \\ \phi(x) = & \cdots \; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \cdots \end{array}$$

FIGURE A.2: The construction for $\mathbb{Z}$.

3) Let the generators of F_k be $S = \{g_1, \ldots, g_k\}$ and let $\Lambda_1 = \{id, g_1, \ldots, g_k, g_1^{-1}, \ldots, g_k^{-1}\}$. Consider the alphabet:

$$\mathcal{A} = \{1, d_1, \ldots, d_k, d_1^{-1}, \cdots, d_k^{-1}\}.$$

Take also the function $\varphi : \Lambda_1 \to \mathcal{A}$ such that $\varphi(id) = 1$ and $\forall 1 \leq i \leq d$ $\varphi(g_i) = d_i$ and $\varphi(g_i^{-1}) = d_i^{-1}$. Consider the finite set of forbidden patterns $\mathcal{F}$ such that every pattern with support Λ_1 which has the symbol 1 in id is forbidden unless each element of Λ_1 has associated it's image under φ. For every $g \in \{g_1, \ldots, g_k, g_1^{-1}, \ldots, g_k^{-1}\}$ we as well forbid every pattern with support $\Lambda_1 \setminus \{g^{-1}\}$ which has the symbol $\varphi(g)$ in id unless each element of $\Lambda_1 \setminus \{id, g^{-1}\}$ has associated it's image under φ. With these rules in hand, setting the symbol 1 as the image of an element of the group immediately fixes the whole point and thus gives an SFT extension to $X_{\leq 1}$ by projecting each d_i and $-d_i$ to 0. The case of $\mathbb{Z}$ can be seen a special case in this proof when $k = 1$. Figure A.3 shows this for F_2.

4) This property can be seen as a direct consequence of proposition 2.4 1) by taking the short exact sequence:

$$1 \to G_1 \to G_1 \oplus G_2 \to G_2 \to 1.$$

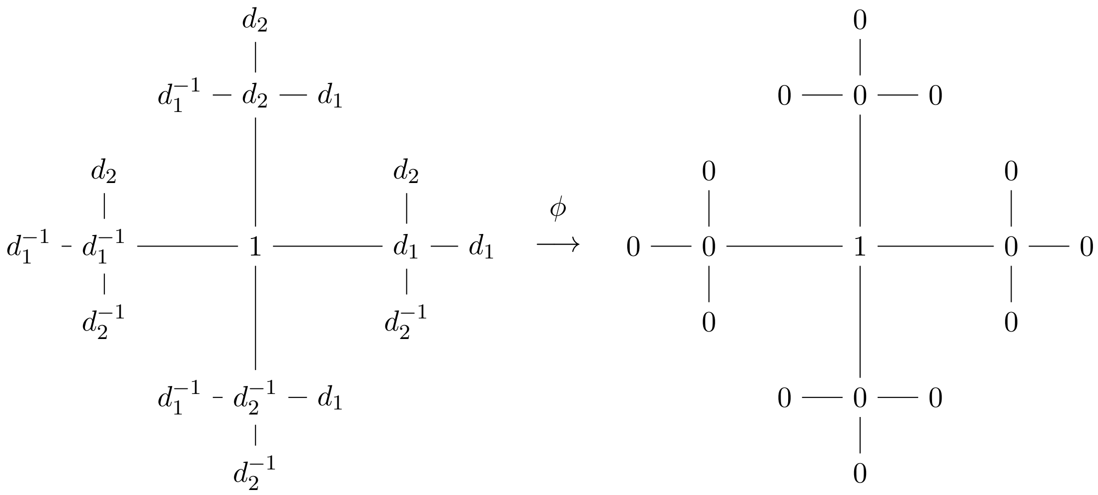


FIGURE A.3: The construction in the case of the free group on two elements, only the elements in Λ_2 are shown.

With the injection and the projection into the second coordinate. A direct proof can be done by considering two presentations for the groups G_1 and G_2 given by a finite number of generators, taking SFT extensions $X_{\mathcal{F}_1} \subseteq \mathcal{A}_1^{G_1}$ and $X_{\mathcal{F}_2} \subseteq \mathcal{A}_2^{G_2}$ given by a 1-block codes defined by local rules Φ_1, Φ_2 and considering the following construction: take $X_{\mathcal{F}} \subseteq (\mathcal{A}_1 \times \mathcal{A}_2)^{G_1 \oplus G_2}$ such that:

$$X_{\mathcal{F}} := \{(x_i, y_j)_{i \in G_1, j \in G_2} | x \in X_{\mathcal{F}_1}, y \in X_{\mathcal{F}_2}\}$$

Obviously the set of forbidden patterns are the ones that forbid that symbols from the first coordinate change when acting under the second group and vice-versa and also ensure that when restricting a point to one coordinate the point we see is in the G_i-subshift corresponding to the other coordinate. That is every rule from the corresponding $\mathcal{F}_i$ is respected. This is a finite amount of rules and thus $X_{\mathcal{F}}$ is an $(G_1 \oplus G_2)$-SFT. By defining a local rule such that $\Phi(a_1, a_2) = 1 \Leftrightarrow \Phi_1(a_1) = 1 \wedge \Phi_2(a_2) = 1$ and 0 otherwise we obtain a 1-block code $\phi = \Phi_\infty$ which maps $X_{\mathcal{F}}$ to $X_{\leq 1}$. Indeed, suppose there are two symbols which get mapped to 1 in a point x, lets say $x_{(id,id)} = (a_1, a_2)$ and $x_{(g_1,g_2)} = (b_1, b_2)$ with $\Phi_1(a_1) = \Phi_1(b_1) = \Phi_2(a_2) = \Phi_2(b_2) = 1$. The forbidden pattern rules imply then that $x_{(id,g_2)} = (c_1, b_2)$ where $\Phi_1(c_1) = 0$, which in turn implies that $x_{(id,id)} = (c_1, c_2)$ which yields a contradiction.

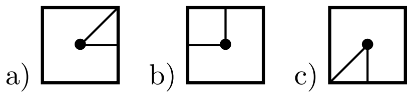
5) Every finitely generated abelian group is the direct sum of a finite number of copies of $\mathbb{Z}$ and a finite group, by using together properties 1, 2 and 5 we obtain this result.

□

Theorem A.3. *The Sierpiński triangle substitution s defined in example 3.1 satisfies the Mozes property from definition 3.7.*

Proof. Let the X_s-substitution s' be defined over an alphabet $\mathcal{A}$. For the sake of simplicity during this proof we will refer to positions $(1,1), (1,2)$ and $(2,2)$ in $R[2,2]$ as 1, 2 and 3 respectively. We will proceed by constructing explicitly the alphabet $\mathcal{A}'$ and the set of finite rules $\mathcal{F}'$ which satisfy the requirements.

Consider the alphabet $\mathcal{A}'$ given by the following three types of tiles:



where each tile carries some extra information which is not shown in the picture. The dot and each of the two segments in each tile carry a tuple belonging to

$$\mathcal{A} \times \{1, 2, 3\} \times \mathcal{A}^3.$$

Where the triple in $\mathcal{A}^3$ is the image under s' of the element in $\mathcal{A}$ (ordered according to the previous simplification of the positions in $R[2, 2]$). It will be represented as:

$$a, i \to \begin{array}{cc} 0 & a_3 \\ a_1 & a_2 \end{array}. \qquad \text{where } i, j \in \{1, 2, 3\} \text{ and } a, a_1, a_2, a_3 \in \mathcal{A}$$

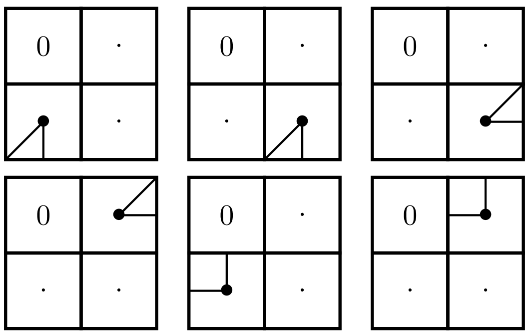
The meaning of such a tuple is the following: it represents the substitution rule $s'(a)$ coming from s' with the additional information that a appears in position i in a previous substitution rule.

The size of the alphabet previously described obviously depends in the particular substitution considered, nevertheless, an upper bound for its size is given by $|\mathcal{A}'| \leq 81|\mathcal{A}|^3$.

Now we proceed to describe the finite set of forbidden patterns $\mathcal{F}'$. We do so by first describing a set of rules and later showing how they can be obtained by forbidden rules. In this description, a single dot $\cdot$ will mean an arbitrary tile from $\mathcal{A}'$, # will mean a number in $\{1, 2, 3\}$ and $*$ will mean a symbol from $\mathcal{A}$.

1. **Structure rule:** The only admissible tilings of $s(1)$ are the ones where tiles of the type a), b) and c) are on positions 1, 2 and 3 respectively. Furthermore, in every such tiling of $s(1)$ the tuple in the dots must be the same for each dot. Also the tuples from lines that match in the border of a tile must coincide.

   It is possible to generate these rules in the following way: The first part of the rule can be easily enforced by checking the neighbors, that is, we forbid every pattern of size 2×2 which has any of the following configurations:



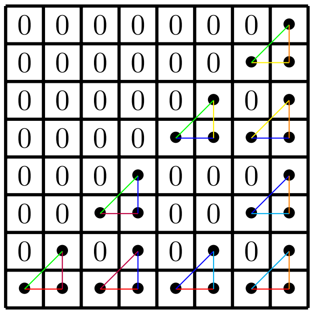


FIGURE A.4: A tiling of $s^3(1)$. Different colors represent different tuples in lines.

The rest of the rules described here can be obtained by further forbidding any 2×2 pattern which does not satisfy the conditions.

2. **Base rule:** Any tiling of $s(1)$ which carries in their dots a rule of the form:

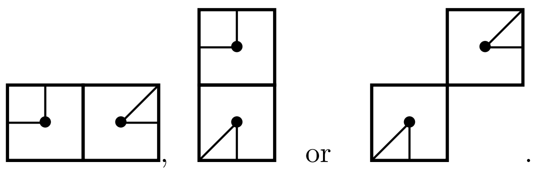
$$a, i \to \begin{array}{cc} & a_3 \\ a_1 & a_2 \end{array}$$

must satisfy the following rule: if second coordinate of the tuple is the number i=1 (respectively 2, 3) then the vertical (resp diagonal, horizontal) line in the triangle must carry a tuple where a must appear in the right hand side in position i, that is:

$$*, \# \to \begin{array}{cc} & a \\ * & * \end{array} \quad \text{supposing } i = 3, \text{ for example}$$

This rule can be obviously obtained in the same way as the rules before.

3. **Pasting rule:** Whenever we encounter patterns of the following shape:



Then we demand that the two lines which have the same orientation must carry the same tuple. Also, the other pair of lines forming an angle must also carry the same tuple between them.

This rule can obviously be enforced by forbidding every pattern with the shapes shown above which does not satisfy the property.

4. **Extension rule:** When encountering patterns as in the last rule: if the tuple shared by the lines forming an angle carries in the second coordinate the number 1 (respectively 2, 3), then if the other two lines which share the orientation are vertical (respectively diagonal, horizontal) then they must carry a tuple which originates from the tuple shared by the lines which make an angle in the same way as in the base rule.

With these rules, we claim that by projecting each tile to the third coordinate in the position given by the type of tile (that is: a) goes to 1, b) to 2 and c) to 3) of the tuple which is held by the black $\bullet$, (this is the local rule Φ) then for every $n \in \mathbb{N}$, every tiling of $s^n(1)$ projects onto $s'^n(a)$ for an $a \in \mathcal{A}$.

Before starting, let $n \in \mathbb{N}$. We will refer to the set of positions of $R[2^n, 2^n]$ given by

$$B_n = \{(1,j), (j,j), (j, 2^n) \text{ for } j \in \{0, \ldots, 2^n\}\}$$

as the n-border and we will call n-skeleton to the set (See figure A.5)

$$S_n = \{(2^{n-1}+1, j), (j+2^{n-1}, j), (j+2^{n-1}, 2^{n-1}+1) \text{ for } j \in \{1, \ldots, 2^{n-1}\}\}.$$

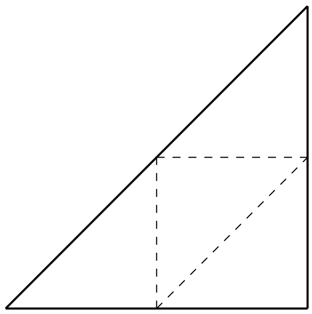


FIGURE A.5: The n-border is given by the black lines and the n-skeleton by the dashed lines.

We proceed by induction. We show the previous claim along with two extra invariants: for any $n \in \mathbb{N}$ the tuple carried by horizontal lines in the lowest part of the border of $s^n(1)$ (respectively by the diagonals or the vertical lines) is the same. Also, for $n \geq 2$, the tuple in the n-skeleton is the same everywhere.

The structure rule ensures that every tiling of $s(1)$ satisfies both the claim and the invariants. The case for $n = 2$ is implied by the fact that any tiling of $s(2)$ is made by pasting together three tilings of $s(1)$, thus, by using the pasting rule we obtain the invariant over the border B_2. The pasting rule also ensures that the rules in the 2-skeleton S_2 must match and hence we have the two invariants. Using the base rule we obtain the claim for $n = 2$.

Suppose both the invariants and the claim are true for $n-1$, using the structure of X_s, that is, that every tiling of $s^n(1)$ is formed by pasting 3 tilings of the previous level, the pasting rules again ensure that the invariants are satisfied for both B_n and S_n. Using the extension rule in the same fashion as the basic rule, we are ensured that each side of the skeleton carries the symbol which generated each one of the three tilings of $s^{n-1}(1)$, and thus the claim is also satisfied by n.

Now consider a tiling $x \in X^s_{\mathcal{F}'}$, as the projection via ϕ of any tiling of $s^n(1)$ yields $s'^n(a)$ for $a \in \mathcal{A}$, then we conclude that $\phi(x) \in X_{s'}$ by definition (every tiling of $X^s_{\mathcal{F}'}$ can be partitioned in either tilings of $s^n(1)$ or patterns consisting only on zeros). In the other sense, it is clear that the construction allows every $s'^n(a)$ for $a \in \mathcal{A}$ which appears in the right side of a substitution rule to appear as the projection of a tiling of $s^n(1)$. As this are the only patterns that appear in $X_{s'}$ for an arbitrary size and they can always be constructed, we conclude that for each $y \in X_{s'}$ there exists a preimage which can be easily obtained by a compacity argument.

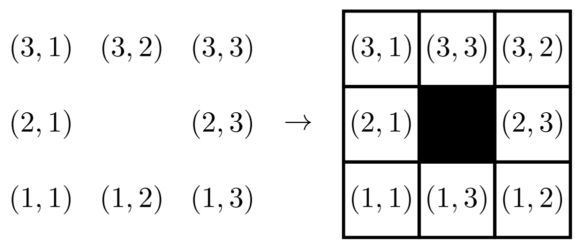
Therefore we conclude that $\phi : X^s_{\mathcal{F}} \twoheadrightarrow X_{s'}$ is a 1-block factor code as demanded by definition 3.7. $\square$

In the previous proof, the reader may note that the information of the right hand side of the substitution was redundant as the symbol of origin already had all of the information given the substitution s. The proof was presented in the previous way in order to make the proof applicable to the more general context of non-deterministic substitutions, where a symbol $a \in \mathcal{A}$ may be substituted in more than one fixed way. Also, presenting the construction in the previous way allows an easier definition of the 1-block code ϕ. These arguments are also applicable to the next theorem.

Theorem A.4. *The Sierpiński carpet from example 3.2 satisfies the Mozes property.*

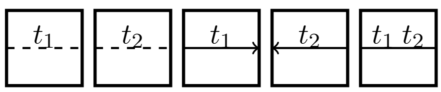
Proof. This proof will yield an explicit extension for any X_s-substitution where s is the Sierpiński carpet. We will proceed by building a hierarchical layer or skeleton which will give the necessary structure in order to communicate the different levels of the substitution s' and therefore generate a foundation for the construction of the layer which carries the substitution information.

We start by defining the alphabet $\mathcal{B}_1$ of the hierarchical layer. Consider the set of valid tuples $T := R[3,3] \setminus \{(2,2)\}$, that is, the positions of $s(1)$ which carry a 1. Each of these tuples will represent an element of $\mathcal{B}_1$ and they will be represented by the following tiles (arranged as the right hand side of the substitution s):

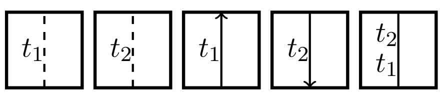


We say that a pair of tuples $(t_1, t_2) \in T^2$ are *horizontally adjacent* if $t_2 = t_1 + (1,0)$ and we say they are *vertically adjacent* if $t_2 = t_1 + (0,1)$.

For every pair of horizontally adjacent tuples (t_1, t_2) we add in $\mathcal{B}_1$ the following tiles:

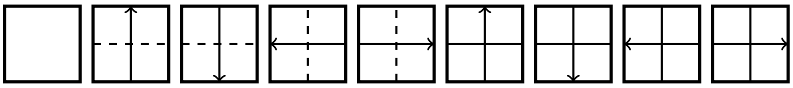


Similarly, for every pair of vertical adjacent tuples (t_1, t_2) we add the following tiles:



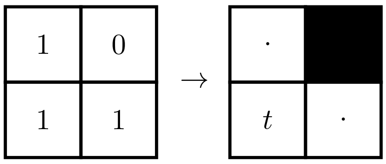
The first two tiles which have dashed lines will be called *basic lines*, the next two tiles which carry arrows will be called *arrow lines*, and the last tile with the segment is called a *middle line*.

For the construction, we will also need a *blank tile*, and a variety of *intersection tiles* where each intersection is between an arrow line and either a basic line or a middle line. Each of the lines in these tiles will carry tuples in the way shown above (these are not shown in the picture).

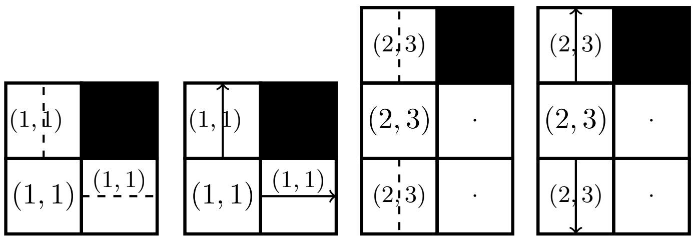


Now we define the set of forbidden patterns for the first layer by means of a set of local rules. The tiles which carry a zero are represented as a black square just for aesthetic reasons:

1. In every pattern of X_s where a 0 is surrounded both in the bottom and in the left positions by a 1, then the position diagonally to the bottom and left of the 0 must carry a tuple, conversely, every tuple tile must have a 0 in the position which is diagonally up and to the right and another tile different from 0 to the right and up:

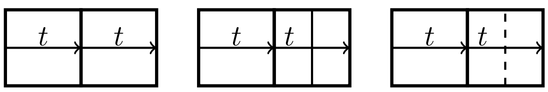


2. Each tuple must be continued in the directions of the two adjacent tuples either by basic lines or by arrow lines carrying the same tuple. We show examples of this rule for tuples $(1, 1)$ and $(2, 3)$.



3. A basic horizontal (respectively vertical) tile can only be continued by another horizontal basic tile carrying the adjacent horizontal tuple. (Either of these basic lines could be part of an intersection tile). Thus the tuples connected by basic lines are at distance 3, see figure A.6.

4. an horizontal (respectively vertical) arrow tile (with no intersections) can only be continued by another horizontal arrow tile carrying the same tuple, or by an intersection tile where the arrow carries the same tuple.



5. A middle tile can only be continued in the direction of the segment either by an identical middle tile, or by an intersection tile. If the connection is with an intersection tile, then the arrow head from the intersection tile must match with the end of the middle tile so that their tuples coincide.

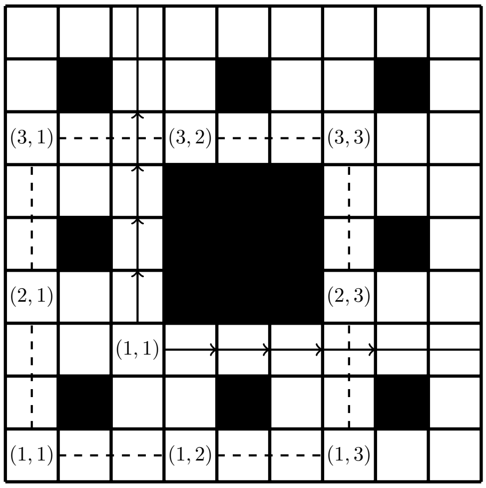
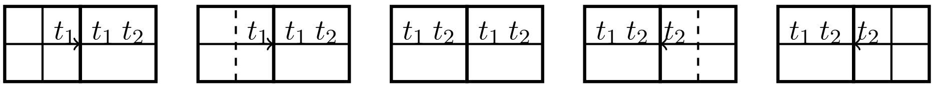


FIGURE A.6: A tiling of $s^2(1)$ using the first layer. The tuples in the lines aren't shown in order to make the picture readable.



6. An intersection line can only be continued in the direction of the arrow by a middle line, and in the opposite direction by an arrow line, all carrying the same tuple as above. the other line (basic or middle) must follow the rules of a normal basic or middle line.

With these rules in hand we have the basic structure ready, and we proceed to build the substitution layer:

The alphabet $\mathcal{B}_2$ will be constructed by using tuples from the set $S := \mathcal{A} \times T \times \mathcal{A}^T \times T$ where any tuple $(a, t_1, (a_t)_{t \in T}, t_2) \in S$ must satisfy that $s'(a) = (a_t)_{t \in T}$. This tuple intends to code the following information: "I carry the symbol a_{t_2} which is generated by substituting the symbol a which in turn appears in position t_1 of another substitution rule".

Each tile of $\mathcal{B}_2$ can carry one, two or three tuples from S depending on the corresponding symbol in the first layer: Every tile carries at least one tuple from S in the background. Also every tuple tile or line carries an extra element of S. Thus tuple tiles and line tiles carry two elements from S and intersection tiles carry three of them (one in the background and two for each line).

Now we define the set of local rules for $\mathcal{B}_2$, which use the structure given by the first layer. Note that this construction is extremely similar to the one shown in theorem A.3:

1. **Structure rule:** Each tiling of $s(1)$ must carry in each of their tiles in the background the same tuple from S except for the last coordinate, where each one of them must match to the position of $s(1)$ that it is tiling, also, the second coordinate of the tuple of S must match to the tuple in the first layer in the position (1,1). This can easily be enforced by using the structure of X_s and forbidding every pattern of size 3×3 which does not satisfy this rule.

2. **Base rule:** In each tiling of $s(1)$, if the tuples of S in the background are $(a, t_1, (a_t)_{t\in T}, \cdot)$, then the tuple of S which goes with the tuple tile t_1 in position (1,1) is of the form: $(b, t_3, (b_t)_{t\in T}, t_1)$ where $b_{t_1} = a$.

3. **Pasting rule:** Any set of 8 tuples which are connected by lines, and the lines that connect them must carry the same tuple from S, except by the last coordinate, which must coincide with the tuple they are carrying (in the case of middle tiles, it must be the smallest one lexicographically)

4. **Extension rule:** When two lines meet in an intersection tile, if the basic or middle lines carries the tuple $(a, t_1, (a_t)_{t\in T}, \cdot)$, then the tuple of the arrow must be of the form $(b, t_3, (b_t)_{t\in T}, t_1)$ where $b_{t_1} = a$.

With these rules in hand, we can proceed to prove the result.

Consider the function Φ which projects every tile by considering the tuple $(a, t_1, (a_t)_{t\in T}, t_2) \in S$ in the background and projecting it to a_{t_2}. We claim that $\phi = \Phi_\infty$ is the desired $1-$block factor map.

In order to prove that, it suffices to show that for any $n \in \mathbb{N}$ the projection by Φ of any tiling of $s^n(1)$ is $s'^n(a)$ for $a \in \mathcal{A}$ which appears at the right hand side of a substitution rule of s' and conversely that every tiling as such can be obtained as a projection of a pattern in the construction. These two facts are easily obtained simultaneously from the fact that for a tiling of $s^n(1)$ if the tuple tile[1] from the first layer in position $(3^{n-1}, 3^{n-1})$, carries the tuple $(a, t_1, (a_t)_{t\in T}, t_2)$ then the image via Φ of the whole block corresponds to $s'^n(a_{t_2})$.

We proceed to show the previous fact by induction, the case for $n = 1$ follows from the first and second rules of the second layer. Now suppose the property is true for every tiling of $s^{n-1}(1)$ and consider a tiling of $s^n(1)$ such that the tuple in the center position described above is $(a, t_1, (a_t)_{t\in T}, t_2)$. The rules from the first layer enforce that in any valid tiling of $s^n(1)$ the tuples originating in the centers of the 8 blocks of shape $s^{n-1}(1)$ must be connected by arrows and middle tiles, and the third rule from the second layer imposes the condition that all these tuples and lines must carry the same tuple from S except for the last coordinate. Using the second rule from the first layer in the tile from the center position we obtain that the information originating in this tuple must extend and eventually intersect the structure formed by the 8 tuple tiles in the center positions of the tilings of $s^{n-1}(1)$. Finally, using the last rule from the second layer in the intersections we obtain that the structure formed by the 8 tuple tiles must carry a tuple of the form: $(a_{t_2}, t_2, (a_t)_{t\in T}, t')$ where t' depends on the tuple carried by the structure of the 8 tuple tiles. using the induction hypothesis we obtain that the image of each of these blocks is $s^{n-1}(s(a_{t_2})_{t'})$ and thus the image of the whole block is $s^{n-1}((s(a_{t_2})_{t'})_{t'\in T}) = s^n(a_{t_2})$. $\square$

[1] This position $(3^{n-1}, 3^{n-1})$ is the upper right corner of the lowest left tiling (it will be called center position) of $s^{n-1}(1)$ composing $s^n(1)$. It must carry a tuple tile according to the first rule of the first layer.

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